* Studies included within the EDUCATIONAL classification are, in most cases, derived from the broad spectrum of scientific and engineering computer applications. Their presentation is oriented toward classroom or laboratory use, and they deal with problems appropriate for chemical, electronics, bio-medical, etc., courses. For the convenience of users in specific industrial fields, however, all EDUCATIONAL studies are cross-referenced in the Applications Library Index under the industrial activity(ies) with which they are most closely associated.

SOLUTION OF MATHIEUS' EQUATION ON THE ANALOG COMPUTER

Many physical systems can be described analytically in terms of the functions of mathematical physics such as Bessel, Mathieu, and Hypergeometric functions. However, an analytical solution of this type is of little value to the investigator unless it can be transformed into usable, numerical results. This transformation often is time consuming and expensive, especially for multiple or trial-and-error computations. For this reason, many of these equations, whose practical solutions are prohibitive, are solved on a general purpose analog computer. The analog computer, which may be programmed with ease, produces continuous, graphical results and allows the analyst to vary parameters in a few seconds for multiple computations, thereby reducing the time and expense required to obtain usable numerical results.

Since the equations of mathematical physics describe the behavior of a great many physical systems, and since the analog computer is a valuable tool in obtaining their solution, it would be advantageous to present the analog computer solution to as many of these equations as possible. Obviously, this is impractical; however, a typical example can be illustrated. The illustration selected is Mathieus' equation whose solution is unique in that it can be stable or unstable, periodic or non-periodic. Mathieus' equation is a practical illustration, also, since it describes the behavior of

1) wave guides
2) moving coil loud-speakers
3) vibrating strings and membranes
4) frequency modulation circuits
5) sinusoidally excited mechanical systems as well as other physical systems. (2)(4)(6)(7)

This Study, then, performed on a desk-top-size PACE® TR-10 general purpose analog computer, describes the solution of Mathieus' equation. The objectives of the study will be threefold: first, to illustrate how Mathieus' equation should be programmed and implemented on the analog computer; second, to show representative results of stable and unstable solutions to this equation; and third, to illustrate the accuracy of the computer by determining points on the stability boundary of the solution and comparing them to the literature values.

Mathematical Model

Mathieus' equation, which is described at length in several references (1, 3, 4, 5, 8), may be represented mathematically in several forms. The form selected for this investigation is

$$\frac{d^2y}{dt^2} + (a - 2q \text{ Cos } 2t)\ y(t) = 0 \tag{1}$$

where y and t are dimensionless dependent and independent variables, respectively. The constants "a" and "q" also are dimensionless. The initial conditions of equation (1) are

$$y\ (o) = 1 \tag{2}$$

$$\left(\frac{dy}{dt}\right)_{t=0} = 0 \tag{3}$$

For simplicity, it has been assumed that

$$a = 2q \tag{4}$$

$$0 \leq a < 5 \tag{5}$$

This restriction, which frequently occurs in practical applications, represents the interdependence of system parameters upon one another and their practical maximum values. If one defines

$$z\ (t) = 1 - \text{Cos } 2t \tag{6}$$

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then equation (1) becomes

$$\frac{d^2y}{dt^2} + a\, z(t)\, y(t) = 0 \qquad (7)$$

A brief discussion of the solution of Mathieus' equation is presented in Appendix C.

Computer Programming †

As shown in Figure 1 (a mathematical block diagram of the system) the function z (t) must be generated and interjected into the simulation to obtain the solution. There are two possible methods of generating z (t). The first method is to use a diode function generator which approximates the function over a fixed range with straight-line segments. This method is unsatisfactory because of the fixed range restriction—±90° is typical—in addition to the errors introduced by the straight-line approximation of the function. It should be noted that special logic circuitry can be programmed in conjunction with the diode function generator to provide a continuous function; however, this only serves to point out the impracticality of this method.

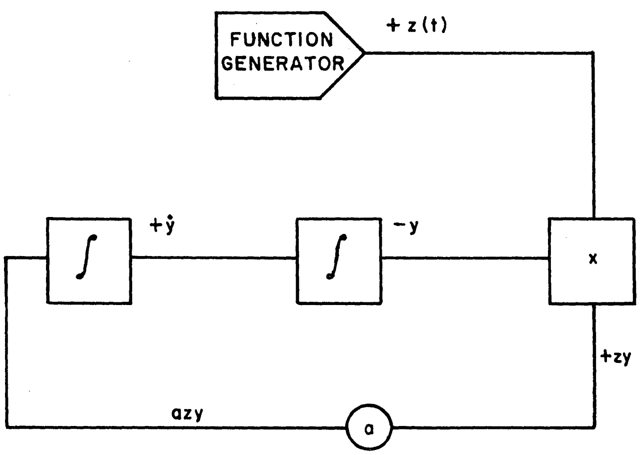

Figure 1. Mathematical Block Diagram

A more accurate and efficient method of generating z (t) is obtained by the solution of a differential equation. The differential equation, which is obtained by differentiating equation (6) twice, is

$$\frac{d^2z}{dt^2} + 4\, z(t) = 4 \qquad (8)$$

which has the initial conditions

$$z(o) = 0 \qquad (9)$$

and

$$\left.\frac{dz}{dt}\right|_{z=0} = 0 \qquad (10)$$

This method is applicable only when the function to be generated is analytic, and a convenient form of its differential equation can be obtained.

The maximum value of z(t) and dz/dt, which was determined while obtaining equation (8), is two. The maximum values of y(t) and dy/dt can be estimated by replacing z(t) in equation (7) by its maximum value to obtain a simplified version of Mathieus' equation, namely

$$\frac{d^2y}{dt^2} + 2a\, y(t) = 0 \qquad (11)$$

The solution to this equation—the equation of an oscillator—using the initial conditions defined by equations (2) and (3) is

$$y(t) = y(o) \cos \omega_n t \qquad (12)$$

where y (o), the initial value of y (t), is unity and the frequency of oscillation, ω_n, is

$$\omega_n = 2a \qquad (13)$$

From equation (12) it is obvious that

$$\frac{dy}{dt} = -\, y(o)\, (\sin \omega_n t)\, \omega_n \qquad (14)$$

At face value, the maximum value of y (t) is unity; however, since some of the solutions of interest in this study are unstable, y (o)—the estimated maximum value of y (t)—was chosen as five to provide a margin of safety. The maximum value of dy/dt then becomes twenty five, since the maximum value of ω_n is

$$\omega_{n_{max}} = \sqrt{2a_{max}} = \sqrt{10} < 5 \qquad (15)$$

and the maximum value of the derivative from equation (14) is y (o) ω_n.

Magnitude scaling is summarized in Table I for a ± 10 volt computer.

† *It is assumed that the reader is familiar with the fundamentals of analog computation.*

TABLE I - MAGNITUDE SCALING SUMMARY

Variable (Dimensionless)	Estimated Maximum Value (Dimensionless)	Scale Factor $\left(\frac{\text{volts}}{\text{Dim. unit}}\right)$	Computer Variable (volts)
y	5	2	$[2y]$
$\dot{y}$	25	2/5	$\left[\frac{2}{5}\dot{y}\right]$
z	2	5	$[5z]$
$\dot{z}$	2	5	$[5\dot{z}]$

The following scaled voltage equations‡ were obtained for z and y

$$\frac{d}{dt}\,[5z] = 10\left(\frac{1}{5}\right)[10] - 10\left(\frac{2}{5}\right)[5z] \qquad (16)$$

$$\frac{d}{dt}\left[\frac{2}{5}\dot{y}\right] = -10\left(\frac{9}{25}\right)\frac{[2y]\;[5z]}{10} \qquad (17)$$

The computer diagram for the simulation is shown in Figure 2 and the potentiometer and amplifier sheets, which include the static check, are shown in Figures 3 and 4. A tabulation of the computing equipment required to perform this simulation is contained in Appendix B.

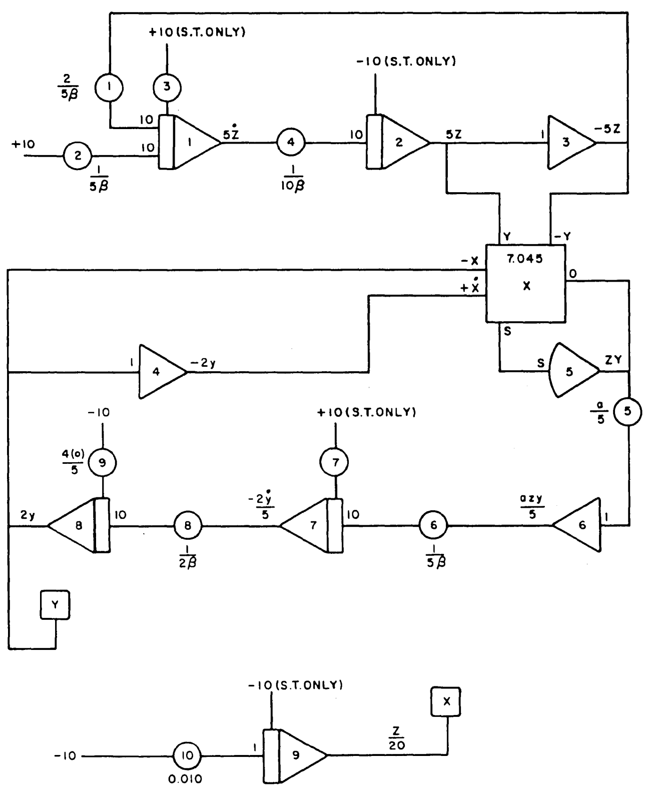

Figure 2. Computer Diagram

‡ [ ] = *reference or computer voltage*
( ) = *potentiometer setting*
10, 1, etc. = input gain

The time scale factor, β, was selected as one half so that as much of the solution as possible could be examined in a reasonable length of time (100 seconds). This selection of β was governed also by the potentiometer settings which could not exceed one.

PROBLEM Mathieus' Eq.

POT NO.	PARAMETER DESCRIPTION	SETTING STATIC CHECK	STATIC CHECK OUTPUT VOLTAGE	SETTING RUN NUMBER 1	NOTES	POT NO.
1	$2/5\beta$	0.800	→	→		1
2	$1/5\beta$	0.400	→	→		2
3	Constant	0.200	→	→		3
4	$1/10\beta$	0.200	→	→		4
5	a/5	0.500			Parametric Variable	5
6	$1/5\beta$	0.200	0.400	→		6
7	Constant	0.300	→	→		7
8	$1/2\beta$	0.300	1.000	→		8
9	y(o)/5	0.800	0.200	→		9
10	Constant	0.010	→	→		10

Figure 3. TR-10 Potentiometer Assignment Sheet

PROBLEM Mathieus' Eq.

AMP NO.	FB	OUTPUT VARIABLE	STATIC CHECK: CALCULATED INTEGRATOR INPUT SUM	CALCULATED OUTPUT	MEASURED INTEGRATOR INPUT SUM	MEASURED OUTPUT	NOTES
1	INT	$-5\dot{z}$	4.00*	-2.00			
2	INT	5z	4.00	10.00			
3	SUM	-5z		-10.00			
4	SUM	-2y		-8.00			
5	HG	zy		8.00			
6	SUM	$\frac{-azy}{5}$		-4.00			
7	INT	$-\frac{2}{5}\dot{y}$	8.00	-3.00			
8	INT	2y	9.00	8.00			
9	INT	t/20	0.10	10.00			

* 10K Feedback in Check Amplifier

Figure 4. TR-10 Amplifier Assignment Sheet

After examining the stability plot of the solution, which is derived in Appendix C and illustrated and tabulated in Appendix B, it was decided that computer runs over the range $0.5 \leq a \leq 4.0$ in 0.5 increments would produce representative results. In addition, trial and error runs to determine the three transition points from stable to unstable solutions in this range must be made.

Results

Typical results of the study are shown in Figures 5, 6, and 7. The results of all runs are summarized in Table II. From this summary, the stable to unstable transition point or stability boundary values of "a" for the system are 0.65, 1.75, and 3.69 over the range of "a" investigated. These points are compared to literature data in Figure 8, which superimposes a = 2q and the computed values of "a" on the stability plot shown in Appendix B.

TABLE II. SUMMARY OF COMPUTER RUNS

Run Number	Parameter Value a	Remarks
1	0.50	Stable
2	0.60	Stable
3	0.62	Stable
4	0.64	Stable
5	0.66	Unstable
6	1.00	Unstable
7	1.50	Unstable
8	1.74	Unstable
9	1.76	Stable
10	1.78	Stable
11	1.80	Stable
12	1.82	Stable
13	2.00	Stable
14	2.50	Stable
15	3.00	Stable
16	3.50	Stable
17	3.66	Stable
18	3.68	Stable
19	3.70	Unstable
20	4.00	Unstable

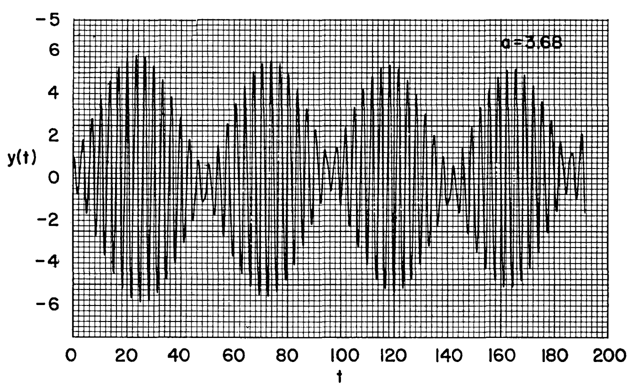

Figure 5. y(t) versus t for α=3.68; solution is stable on the verge of being periodic

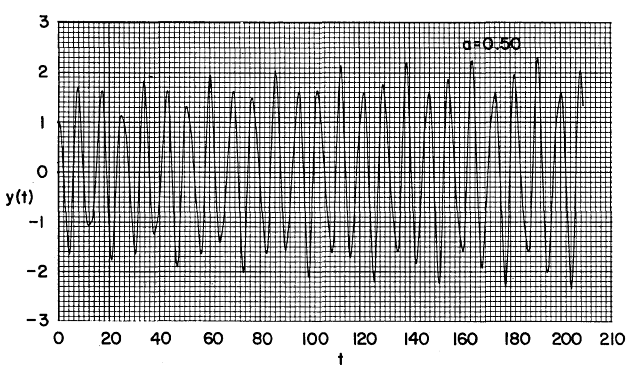

Figure 6. y(t) versus t for α=0.50; solution is stable, non-periodic

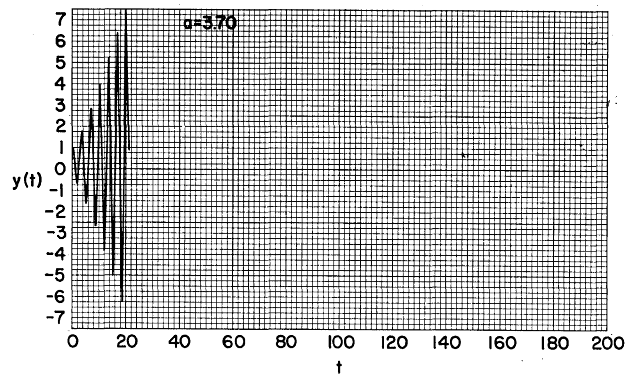

Figure 7. y(t) versus t for α=3.70; solution is unstable

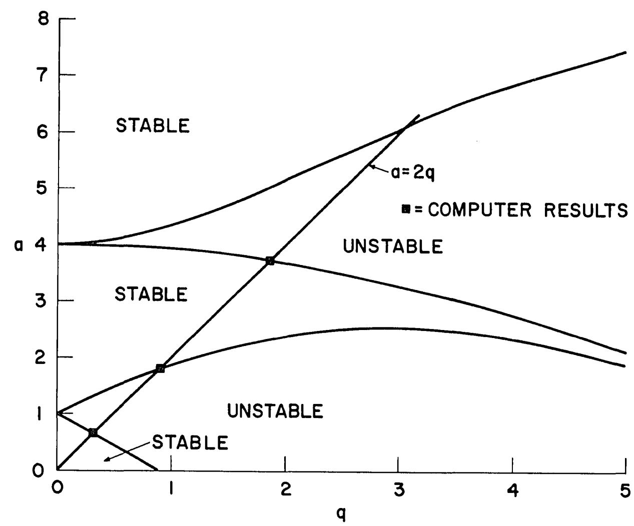

Figure 8. Comparison of Theoretical and Computer Results

Conclusions

The objectives of the study have been achieved. First, the mechanization of Mathieus' equation on the analog computer has been illustrated as well as several noteworthy points regarding function generation. A simple but accurate method of generating an <u>analytic</u> function is from the solution of a differential equation, which generates the function. This technique presumes that a convenient form of the differential equation can be obtained. In the case of periodic or sinusoidal functions this is the most practical method of obtaining a continuous function.

Typical solutions, which are cosine elliptic*, of Mathieus' equation are shown in Table I, Runs 1 through 20 (specifically, Figures 5, 6, and 7).

** Sine elliptic and cosine elliptic solutions of Mathieus' equation are defined in Appendix A.*

These non-periodic solutions behave as expected, as "a" increases so does the frequency of the solution. By consulting the literature (5), it was found that the sine elliptic solutions of Mathieus' equation behave in a similar manner. The time required to obtain these results is trivial (less than one hour) when compared to other computational methods.

The percent error of the three computed stability boundary points compared to the literature values[5] is less than 4%. This error is very small when one considers the error usually associated with the parameters used in scientific and engineering studies.

The most significant result of this study is the fact that the stability boundary points could be determined using the analog computer. This computer application permits a system analyst or design engineer to select parameters and operating conditions for efficient, stable operation of a system with relative ease.

APPENDIX A

TABULATION OF EQUIPMENT

The following major computing components were required to perform this study.

9 Operational Amplifiers

5 Integrator Networks

10 Potentiometers

1 Multiplier

1 X-Y Plotter

APPENDIX B

STABILITY PLOT DATA(5)

q	b_1	a_1	b_2	a_2	b_3
0	1.00	1.00	4.00	4.00	9.00
1	-0.11	1.86	3.92	4.37	9.05
2	-1.39	2.38	3.67	5.17	9.14
3	-2.79	2.52	3.28	6.05	9.22
4	-4.26	2.32	2.75	6.83	9.26
5	-5.79	1.86	2.10	7.45	9.24

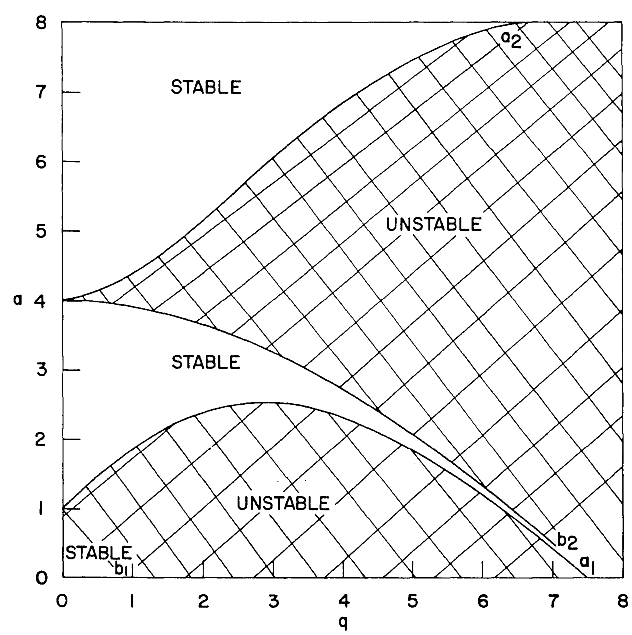

Figure 9. Stability Diagram

APPENDIX C

ANALYTICAL CONSIDERATIONS OF MATHIEUS' EQUATION

One of the more common representations of Mathieus' equation is

$$\frac{d^2 y(t)}{dt^2} + (a - 2q \cos 2t)\, y(t) = 0 \qquad (1)$$

where "a" and "q" are constants. The stable solutions to this equation, which are oscillatory, may be periodic or non-periodic. Fortunately, we need only consider those solutions which are periodic, since the relationships between "a" and "q" on an "a versus q" plot for the periodic solutions forms the stability boundaries of the solution(5). The odd and even (sin or cos) solutions to equation (1) are called Mathieu functions, which are defined in power series as*

$$ce_m(t, q) = \cos m t + q\, c_1(t) + q^2\, c_2(t) + \cdots \qquad (2)$$

and

$$se_m(t, q) = \sin m t + q\, s_1(t) + q^2\, s_2(t) + \cdots \qquad (3)$$

where m denotes the order of the function. The "characteristic numbers" of ce_m and se_m are denoted by a_m and b_m respectively (a_m and b_m are actually a in equation 1) and are related to "q" by a power series

$$a_m, b_m = m^2 + \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 + \cdots \qquad (4)$$

whose coefficients depend on the order and type of solution.

An "even" solution to equation (1)

$$y(t) = ce_m(t, q) \qquad (5)$$

is obtained when

$$y(o) = 1 \qquad (6)$$

and

$$\left(\frac{dy}{dt}\right)_{t=0} = 0 \qquad (7)$$

while an "odd" solution

$$y(t) = se_m(t,q) \qquad (8)$$

is obtained when

$$y(o) = 0 \qquad (9)$$

and

$$\left(\frac{dy}{dy}\right)_{t=0} = m \qquad (10)$$

A general solution to equation (1) is a linear combination of ce_m and se_m

$$y(t,q) = A\, ce_m(t,q) + B\, se_m(t,q) \qquad (11)$$

where A and B are constants of integration.

The coefficients of equation (4) are determined by substituting either equation (2) or (3) and equation (4) into equation (1) and solving for the unknown c or s terms. The α coefficients of equation (4) are then selected to yield a periodic solution. For example, if m were unity

$$y(t) = \cos t + q\, c_1(t) + q^2\, c_2(t) + q^3\, c_3(t) + \cdots \qquad (12)$$

$$\frac{d^2 y}{dt^2} = -\cos t + q \frac{d^2 c_1}{dt^2} + q^2 \frac{d^2 c_2}{dt^2} + q^3 \frac{d^2 c_3}{dt^2} + \cdots \qquad (13)$$

$$ay(t) = \cos t + q[c_1(t) + \alpha_1 \cos t] + q^2[c_2(t) + \alpha_1 c_1(t) + \alpha_2 \cos t] + \cdots \qquad (14)$$

and

$$-(2q \cos 2t) = -q(\cos t + \cos 3t) - 2q^2\, c_1(t) \cos 2t - \cdots \qquad (15)$$

Collecting like powers of q yields

$$q^o \quad \cos t - \cos t = 0 \qquad (16)$$

* *se and ce stand for sine elliptic and cosine elliptic.*

$$q \frac{d^2 c_1}{dt^2} + c_1(t) - \cos 3t + (\alpha_1 - 1) \cos t = 0 \quad (17)$$

and

$$q^2 \frac{d^2 c_2}{dt^2} + c_2(t) + \alpha_1 c_1(t) - 2c_1(t) \cos 2t + \alpha_2 \cos t = 0 \quad (18)$$

Since the particular integral corresponding to $(\alpha_1 - 1) \cos t$ is the non-periodic function $1/2 (1 - \alpha_1) t \sin t$, $_1$ is chosen as unity. Therefore,

$$c_1(t) = -\frac{1}{8} \cos 3t \quad (19)$$

satisfies equation (17). This method may now be repeated to determine as many terms of the series as desired. The results of the example are

$$ce_1(t, q) = \cos t - \frac{1}{8} q \cos 3t + \frac{1}{64} q^2 (-\cos 3t + \frac{1}{3} \cos 5t) + \cdots \quad (20)$$

and

$$a_1 = 1 + q - \frac{1}{8} q^2 - \frac{1}{64} q^3 - \frac{1}{1536} q^4 + \cdots \quad (21)$$

If the above procedure is repeated for several integral values of "m" a stability plot, which is shown in Figure 9, can be obtained. The data for this plot is tabulated in Appendix B. Only the first quadrant of the stability plot is considered in this study. However, it should be noted that the second quadrant is a minor image of the first quadrant.

LIST OF SYMBOLS

a	=	Parameter	Dimensionless
a_m	=	Characteristic number of ce_m	Dimensionless
b_m	=	Characteristic number of se_m	Dimensionless
m	=	Order of Mathieu function	Dimensionless
q	=	Parameter	Dimensionless
y	=	Dependent variable	Dimensionless
z	=	Frequency variable	Dimensionless
A	=	Constant of integration	Dimensionless
B	=	Constant of integration	Dimensionless
β	=	Time scale factor	Seconds / Dimensionless Unit
ce_m	=	Mathieu function (cosine elliptic)	Dimensionless
se_m	=	Mathieu function (sine elliptic)	Dimensionless

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