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Diffusion Attenuation Part II

Abstract: The amount of diffusion attenuation has been computed as a function of frequency for the case of uniform electric field. Application to drift transistors is discussed.

Introduction

This paper contains the results of a computation of diffusion attenuation and phase change as a function of frequency for the case of uniform electric field. These results are cast in a form intended to maximize their usefulness to those interested in the complex transfer efficiency of drift transistors. Part I of this paper describes the physical ideas in detail.

The category of problems connected with injected-carrier flow in a uniform electric field, subject also to diffusion, has a long history in the literature. The question of the attenuation of a minority-carrier pulse in a drift-mobility experiment,[1] for example, is a problem in the category. The case of harmonic time dependence of injected-carrier density, as considered here, has been treated by Kroemer.[2] The present computational results, however, are more exact than previous ones. These results will be expressed graphically in some detail, with indication of their possible application to the determination of transistor parameters.

Theory

The discussion of the attenuation of carrier flow across the base must be founded upon the equation of continuity, which, in the presence of a uniform electric field, takes the form[3]

$$p''-p'=v^{-1}\dot{p}, \tag{1}$$

where p is the injected (assumed positive) carrier concentration, and

$$v=\mu E, \qquad \lambda=V_T/E, \tag{2}$$

where μ is the mobility; E, the electric field; V_T, thermal voltage (kT/e). One recovers the usual form of the equation by multiplying by v, and noting that $D=v\lambda$, where D is the diffusion constant.

We are concerned here with the case of harmonic time dependence. Since the differential equation is linear we should be able to obtain an exponential solution, or in real form,

$$p=\cos k(x-ut)e^{-ax}, \tag{3}$$

where k, u, and a are constants. This is an attenuated harmonic wave propagating with phase velocity u. With diffusion present we should expect both signs of u to be possible, since signals may be propagated by diffusion against an electric field. Correspondingly, a will have two possible values. In fact, substituting this expression in the differential equation, we find, on collecting cosine terms,

$$\lambda(a^2-k^2)+a=0,$$

yielding

$$a=\frac{-1\pm\sqrt{1+4\lambda^2k^2}}{2\lambda}. \tag{4}$$

On collecting sine terms we obtain

$$u=v(1+2a\lambda)=\pm v\sqrt{1+4\lambda^2k^2}. \tag{5}$$

The plus sign is, of course, correlated with signals which are propagated in the direction of the field. It will be convenient to introduce a new symbol b as the abbreviation of the positive square root

$$b=+\sqrt{1+4\lambda^2k^2}. \tag{6}$$

A solution for p which vanishes at a point $x=W$ can be constructed by proper combination of forward and reverse signals as follows:

$$p=e^{\frac{x}{2\lambda}}\left[\cos k(x-W-vbt)e^{-\frac{bx}{2\lambda}}\right.$$
$$\left.-\cos k(x-W+vbt)e^{\frac{bx}{2\lambda}}e^{-\frac{W}{\lambda}}\right]. \quad (7)$$

This boundary condition is chosen to achieve continuity between the case of pure diffusion, which requires some such boundary condition in order that a current be possible at low frequencies, and the case of large electric fields, for which the boundary condition at $x=W$ is largely irrelevant. The desirability of imposing this boundary condition, equivalent to letting $v=\infty$ at $x=W$, led to our discussing the problem here in terms of the concentration p rather than directly in terms of the current j, as in the preceding paper.

We must now compute the current, however. Our aim is to find the attenuation and phase shift of the current over the distance W. In accordance with transistor terminology, the current at $x=0$ will be called the emitter current, j_ε and that at $x=W$ the collector current, j_c. We can certainly write

$$j_\varepsilon=A\cos\omega t-B\sin\omega t, \qquad (\text{at } x=0) \quad (8a)$$

$$j_c=C\cos\omega t-D\sin\omega t, \qquad (\text{at } x=W) \quad (8b)$$

where A, B, C, and D are real constants (in time), three of which are independent. The attenuation factor, or transfer efficiency, β can be written in terms of these constants as

$$\beta=\left(\frac{C^2+D^2}{A^2+B^2}\right)^{1/2}, \quad (9)$$

and the phase difference θ between j_ε and j_c is

$$\theta=\tan^{-1}\frac{BC-AD}{AC-BD}. \quad (10)$$

There is little point in carrying through explicitly the trivial but laborious task of determining these parameters. Expressions for them follow straightforwardly from the definition of the current, $j=ev(p-\lambda p')$. Before writing these expressions we define several quantities which furnish a convenient characterization of the physical situation.

The region between $x=0$ and $x=W$, the "base" in transistor applications, is completely defined by two parameters, the length W and the potential difference $\Delta V=EW$. It is defined as well by two more convenient and mathematically significant parameters

$$r=\Delta V/V_T \quad (11)$$

and

$$\omega_D=2D/W^2. \quad (12)$$

The latter is the "cutoff frequency" of a diffusion transistor of base width W.

With parameters defining the physical situation specified, we still have the possibility of redefining the variable of the problem. Instead of the frequency ω we shall employ the angle

$$\phi=kW. \quad (13)$$

The quantity ϕ has the meaning of θ_0 in *I*; i.e., it is the phase difference between emitter and collector currents in the absence of diffusion. It is hoped that the difference in significance of the symbol ϕ in Parts I and II will not cause confusion.

The advantage of taking the quantities r, ω_D, and ϕ as fundamental lies in the fact that all other quantities may easily be computed from them, and also in the fact that A, B, C, and D are functions of r and ϕ alone. The quantity ω_D enters the problem in fact only as a sort of frequency-normalizing factor. The relation between ω and ϕ is

$$\omega/\omega_D=\tfrac{1}{2}rb\phi, \quad (14)$$

where b, as defined previously, appears as

$$b=\sqrt{1+\left(\frac{2\phi}{r}\right)^2}. \quad (15)$$

A final definition is a mere abbreviation

$$\psi=br/2. \quad (16)$$

In terms of ϕ, r, and the quantities b and ψ, which are functions of ϕ and r, we find

$$A=(b\cosh\psi+\sinh\psi)\cos\phi-\frac{2\phi}{r}\sinh\psi\sin\phi, \quad (17a)$$

$$B=(b\sinh\psi+\cosh\psi)\sin\phi+\frac{2\phi}{r}\cosh\psi\cos\phi, \quad (17b)$$

$$C=be^{r/2}, \quad (17c)$$

$$D=\frac{2\phi}{r}e^{r/2}. \quad (17d)$$

All functional relationships may now be computed rather easily by fixing values of r and ϕ and computing all other quantities from these. This applies, for example, to the interesting function $\beta(\theta)$.

It should be noted that use has been made of the arbitrary scale factor contained in the constants A, B, C, and D to normalize them at will. This freedom will also be tacitly employed in the following section, wherein certain interesting special cases will be examined.

• *Case 1:* $r\to\infty$

Discarding negative exponentials and factors multiplied by r^{-1} (except in exponentials) we obtain

$$A\doteq e^{br/2}\cos\phi$$

$$B\doteq e^{br/2}\sin\phi$$

$$C\doteq e^{r/2}$$

$$D\doteq 0.$$

Thus

$$\beta\doteq e^{(1-b)r/2}$$

$$\theta=\phi. \quad (18a)$$

Approximating now in the exponential,

$$(1-b) \doteq -\frac{2\phi^2}{r^2} \doteq -\frac{2\theta^2}{r^2},$$

and hence

$$\beta \doteq e^{-\theta^2/r}$$

$$\left(\omega \doteq \frac{\theta r}{2}\,\omega_D\right). \tag{18b}$$

Equation (18b) was derived as an asymptotic formula in (I).

• *Case 2: $r \to 0$*

After multiplying through by $r/2\phi$ and discarding terms containing a factor r (and noting

$$\lim_{r\to 0} \frac{br}{2\phi} = 1, \qquad \lim_{r\to 0} \psi = \phi)$$

we obtain (except for a scale factor)

$$A = \cosh\phi\cos\phi - \sinh\phi\sin\phi$$

$$B = \sinh\phi\sin\phi + \cosh\phi\cos\phi$$

$$C = 1$$

$$D = 1.$$

Thus

$$\theta = \tan^{-1}(\tanh\phi\tan\phi) \tag{19a}$$

$$\beta = [\cosh^2\phi\cos^2\phi + \sinh^2\phi\sin^2\phi]^{-\frac{1}{2}} \tag{19b}$$

$$(\omega = \phi^2\omega_D).$$

These are the usual formulas for the ungraded base. The formulas for β and θ are often combined in the prescription[4]

$$\beta_{(\text{complex})} = \operatorname{sech}[\phi(1+i)] = \operatorname{sech}\sqrt{2i\frac{\omega}{\omega_D}}\,.$$

• *Case 3: $\phi \to \infty$ ($\omega \to \infty$)*

After multiplying by $r/2\phi$ and noting $\psi \to \phi$, we find

$$A = \cosh\phi\cos\phi - \sinh\phi\sin\phi$$

$$B = \sinh\phi\sin\phi + \cosh\phi\cos\phi$$

$$C = e^{r/2}$$

$$D = e^{r/2}.$$

This result is identical to that for the ungraded case, except that the magnitude of β is increased by the factor $e^{r/2}$.

To complete the analysis of this case, we should let ϕ approach infinity in the exponential terms also. This operation yields

$$\theta = \phi \tag{20a}$$

$$\beta = 2\exp\left(-\sqrt{\frac{\omega}{\omega_D}} + \frac{r}{2}\right) = 2\exp\left(-\theta + \frac{r}{2}\right). \tag{20b}$$

• *Case 4: $r > 4$*

The negative exponentials can be discarded in this case with an error of about one percent or less. However, all other factors must be retained. The simplification yields

$$\theta = \phi - \tan^{-1}\left[\frac{2\phi}{r}\left(1+b+2\left(\frac{2\phi}{r}\right)^2\right)^{-1}\right] \tag{21a}$$

$$\beta^2 = 2\,\frac{1+2\left(\frac{2\phi}{r}\right)^2}{1+b+\left(\frac{2\phi}{r}\right)^2}\,e^{-(b-1)r}. \tag{21b}$$

The difference between θ and ϕ has an upper limit in the neighborhood of $\pi/8$, and approaches zero at both high and low frequencies. These expressions combine to yield the asymptotic formula of Case 1 for $2\theta/r \ll 1$, and the formula of Case 3 for $2\theta/r \gg 1$. The parametric form cannot be avoided for intermediate values of θ. The errors involved in using the asymptotic formulas are (respectively) of order $(\theta/r)^2$ and $(r/\theta)^2$.

In the case of the first asymptotic formula the error in a given range of $|\beta|$ varies inversely as r, and is less than 3% for $r=16$, $0 \leqslant |\beta| < \frac{1}{2}$.

Results

The transfer efficiency β has been computed from the previous equations, and the results are shown in Figs. 1 and 2. Each Figure shows a family of curves plotted for various values of the normalized drift potential $r \equiv \Delta V/V_T$, the ratio of the potential difference across the base to thermal voltage. The variable ξ is the normalized frequency, $\xi \equiv \omega/\omega_D$, where $\omega_D = 2D/W^2$. Recall that ω_D is the cutoff frequency of a corresponding diffusion transistor with the same base width.

Figure 1 is a polar plot of the magnitude of the transfer efficiency β as a function of the phase angle. The changing value of ξ is indicated as a parameter of the curve. Figure 2 displays the relation between β and ξ directly. No curves are given for values of r greater than 16, since the approximate formulas for large r given in the text are very nearly exact in the omitted range.

Unless the transfer efficiency is the only frequency-dependent transistor characteristic over an extended frequency range, our method of determining transistor parameters, in order to be applicable, must be supplemented.[5] Furthermore, its applicability in principle depends upon the structural type of the transistor in question. However, it is not unreasonable to expect in some cases a situation allowing direct applicability.

The existence of such a favorable situation can, of course, be ruled out if the shape of the experimentally measured current gain α does not conform to that calculated for β. Where such conformance is experimentally determined, at least for low frequencies, effective values for the drift potential ΔV and the base width W may be determined by identifying β and α/α_0, where α_0 is the current gain at zero frequency. A convenient procedure for determining these parameters is sketched below.

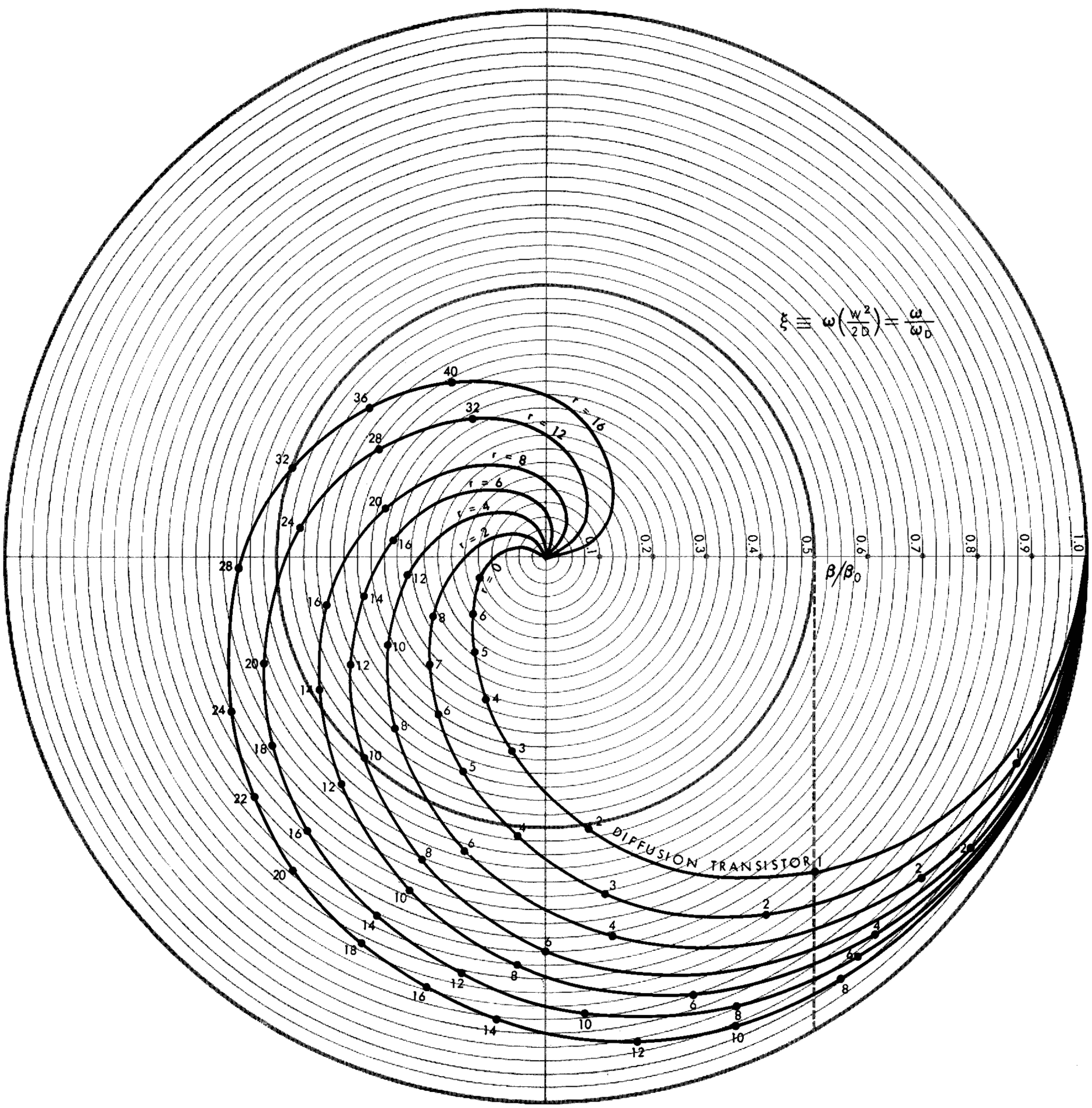


Figure 1 **Complex transfer efficiency of a drift transistor.**
All numbers on curves are values of ξ. Curves beyond highest marked values of ξ are projected.

Determination of effective values of ΔV and W

If the above analysis is found to be applicable in a given case, two measurements are sufficient to determine the two structure parameters ΔV and W. One may choose, for example, to measure the frequency at which the complex current gain α takes on a predetermined value. Two such frequencies, which are often employed to characterize transistors, are defined by the relations:

$$|\alpha(\omega_\alpha)| = \frac{1}{\sqrt{2}}\alpha_0$$

$$Re\alpha(\omega_b) = \frac{1}{2}\alpha_0.$$

It is evident that ω_a is the cutoff frequency for the current gain (normalized to its value at zero frequency) while ω_b is the frequency at which the grounded emitter current gain (normalized to infinity at zero frequency) has dropped to unity.

The following procedure has been facilitated by some supplementary Figures, to which reference will be made. Having measured ω_a and ω_b, one may determine

(i) $r \equiv \Delta V/V_T$ from Fig. 3,

(ii) $\xi_b \equiv \omega_b/\omega_D$ from Fig. 4,

(iii) W from Fig. 5.

For reference a curve of ω_a/ω_D is also plotted in Fig. 4, and compared with Kroemer's approximate formula for high drift fields.

Figure 6 contains a graphical comparison of the asymptotic expression for diffusion attenuation for large r with the exact function.

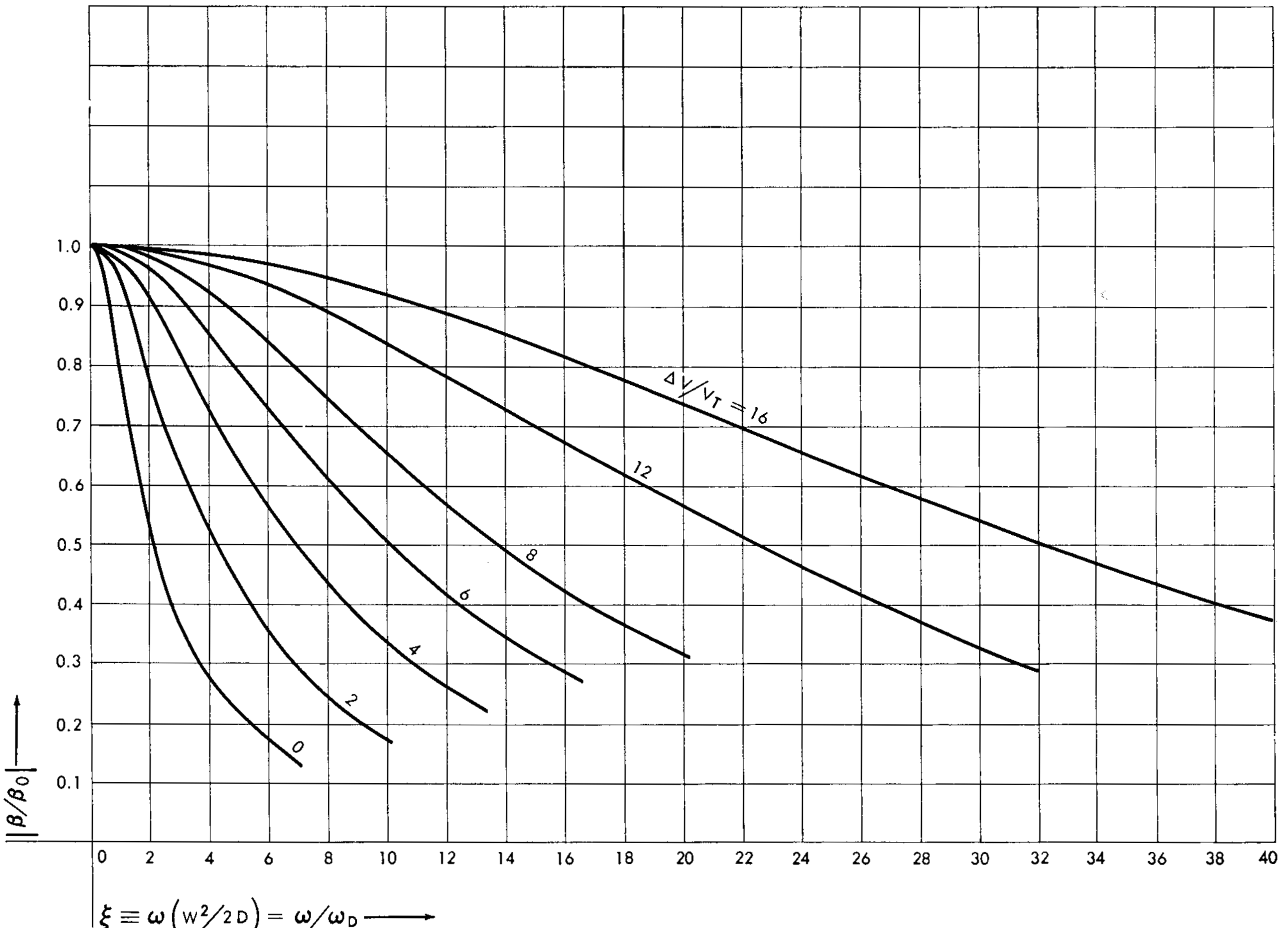
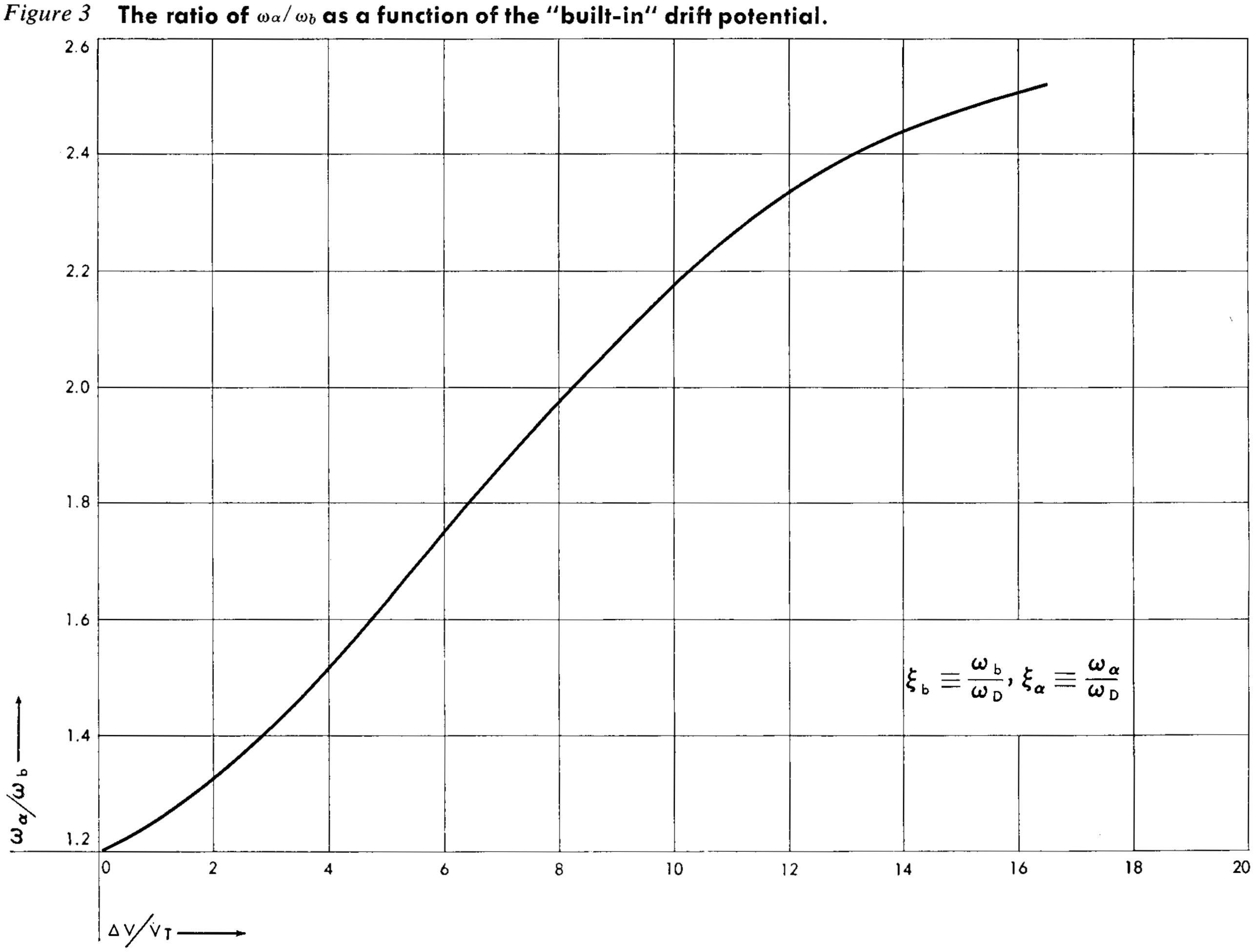


Figure 2 **Variation of the magnitude of the transfer efficiency of a drift transistor with (normalized) frequency.**

Figure 3 **The ratio of ω_α/ω_b as a function of the "built-in" drift potential.**



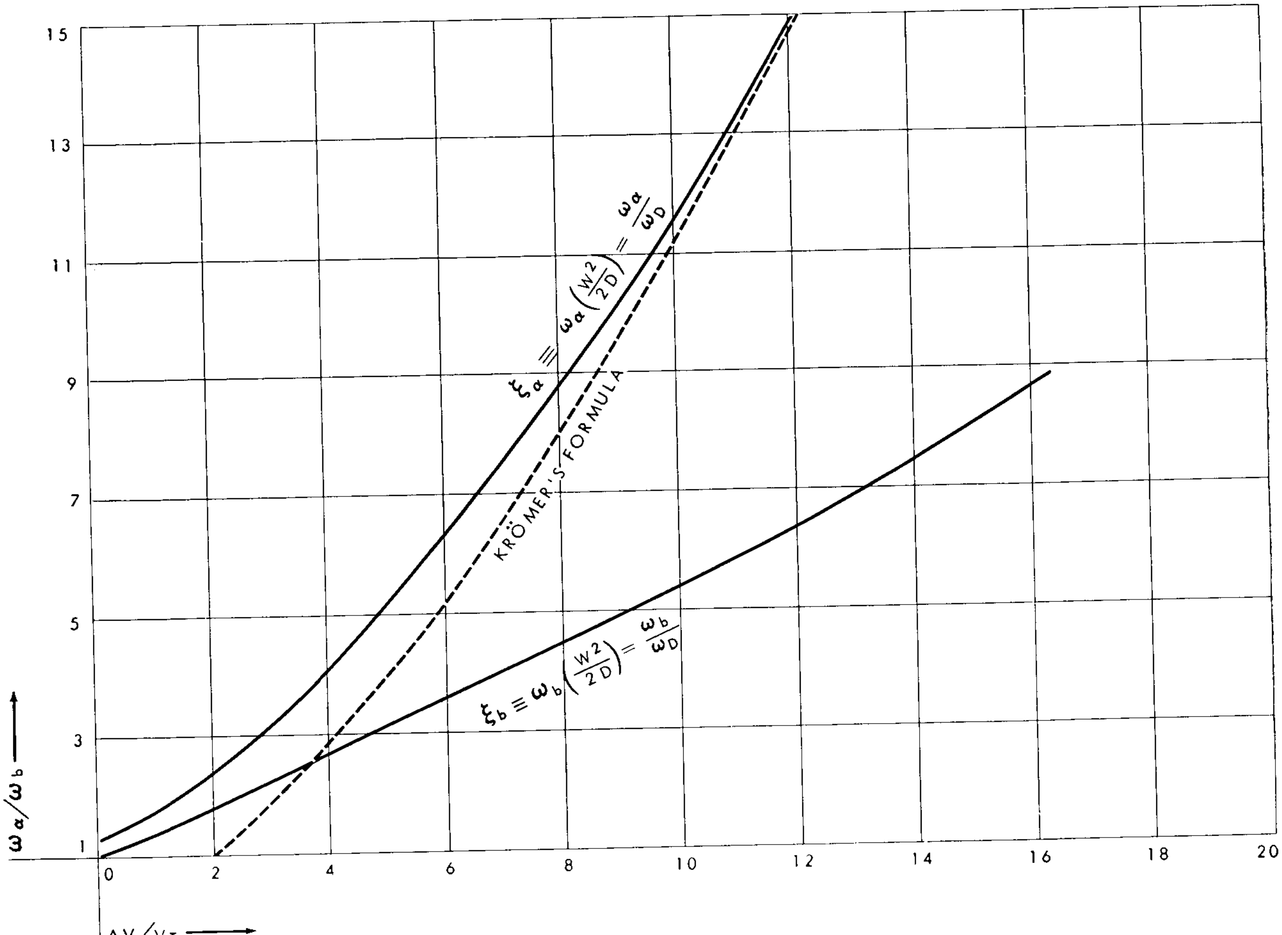
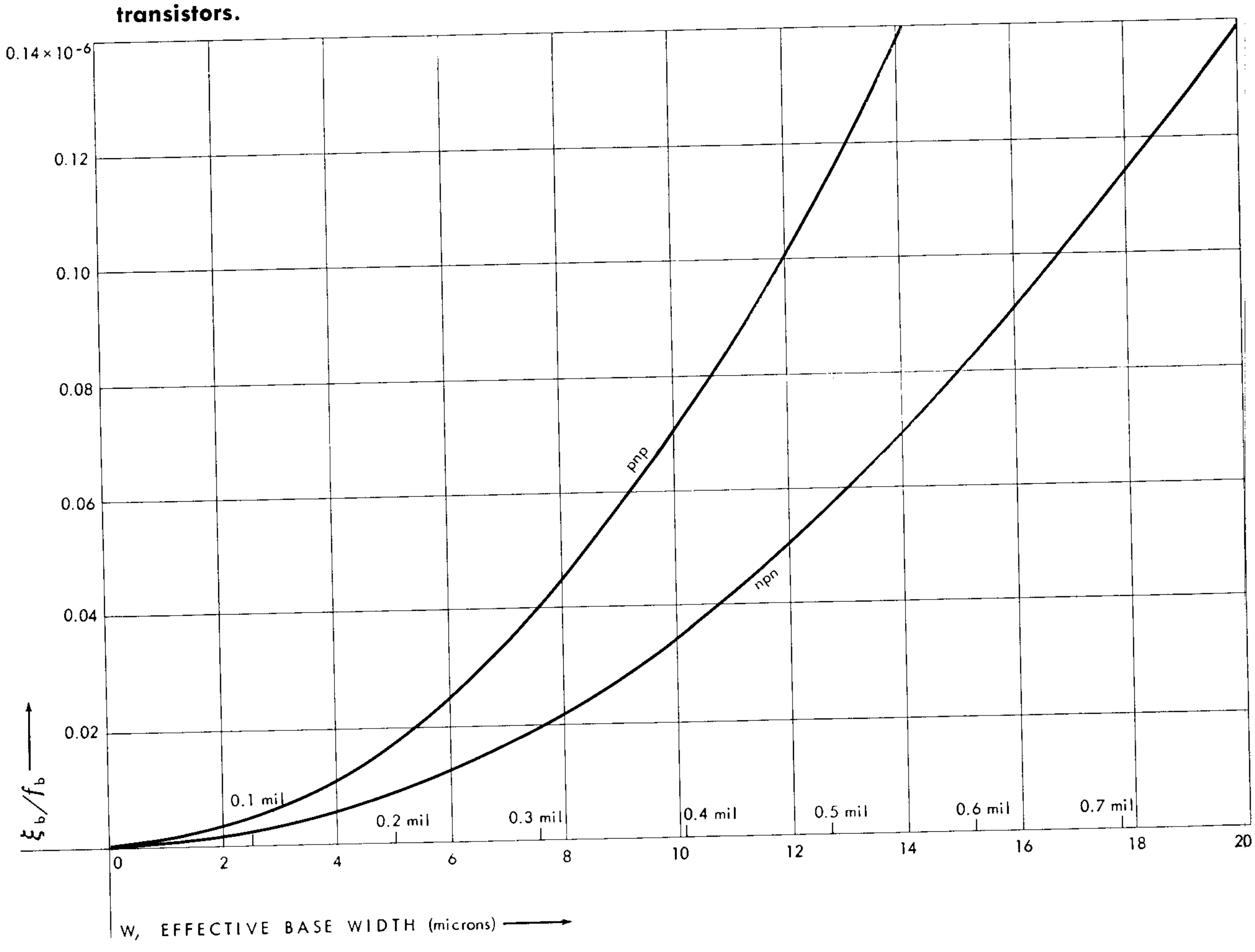


Figure 4 **(Normalized) cutoff frequency as a function of the "built-in" drift potential.**

Figure 5 **The relationship between the effective base width and cutoff frequencies of germanium drift transistors.**



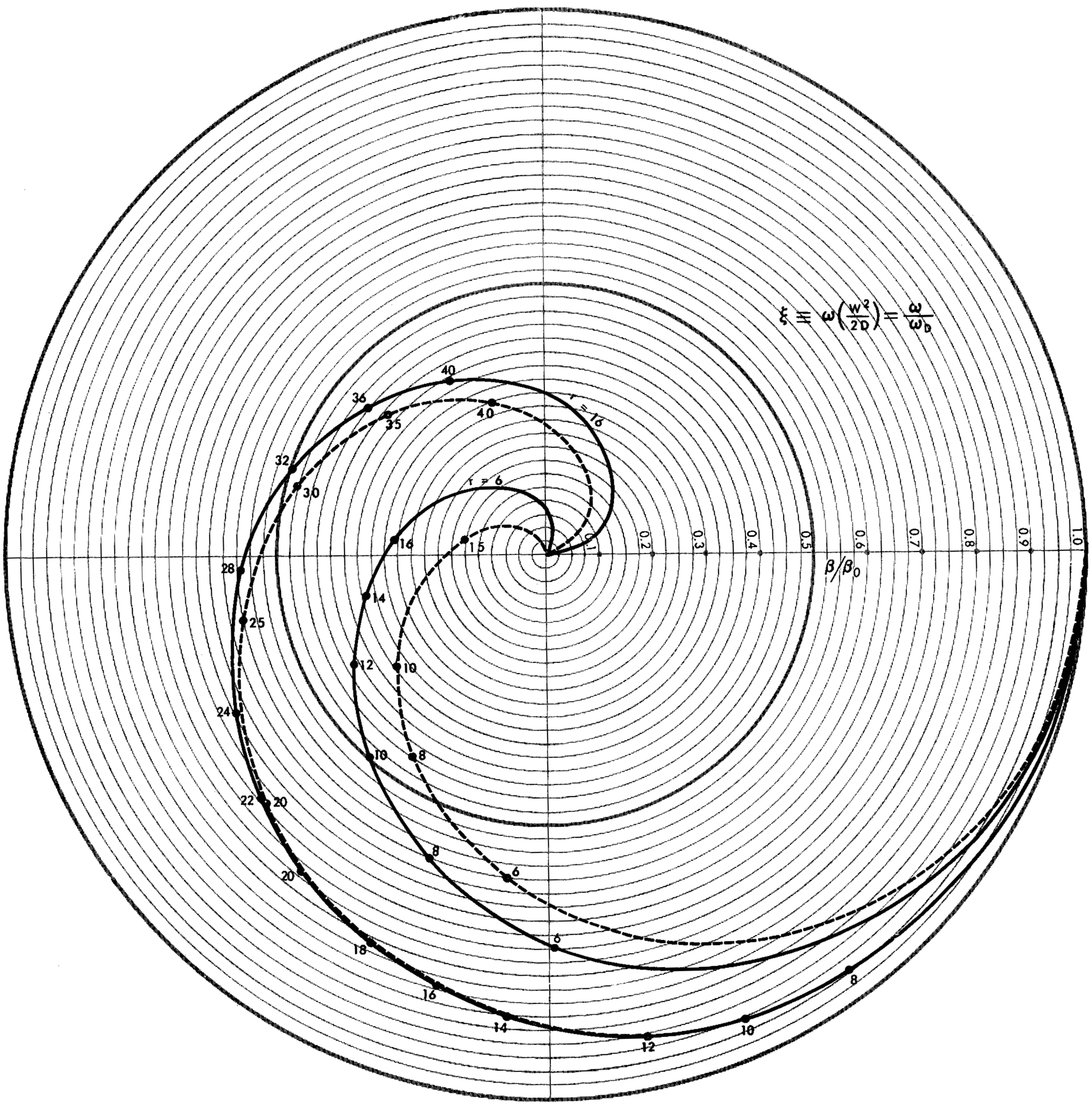


Figure 6 **Showing the relation between asymptotic (dashed line) and exact (solid line) functions $\beta(\theta)$.** *All numbers on curves are values of ξ.*

Appendix: Inclusion of recombination

Since carriers are not conserved, the equation of continuity becomes

$$\lambda p'' - p' = v^{-1}(\dot{p} + \tau^{-1}p),$$

where $\tau^{-1}p$ represents the local rate of removal of carriers. No great complication is introduced if τ is assumed to be a constant independent of carrier concentration and position. The only change resulting from including recombination occurs in the square root term in the expression for a and u as indicated below

$$\sqrt{1+4\lambda^2k^2} \rightarrow \sqrt{1+4(v\tau)^{-1}\lambda+4\lambda^2k^2}\ .$$

Thus in the parametric form of the general expressions for β and θ we need only change the definition of b:

$$b \equiv \sqrt{g+(2\phi/r)^2}\ ,$$

where $g = 1+8/\omega_D\tau r^2$, a constant independent of ϕ. When r is large (special case #1), the attenuation due to recombination is constant; namely, $\exp(-t/\tau)$, where t is the time required for propagation of a signal across the base: $t = W/v = 2/\omega_D r$. Thus for this case recombination is completely taken into account by the normalization of β.

References

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Revised manuscript received June 26, 1958