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# Random-walk Model of Stream Network Development*

**Abstract:** A random-walk, headward-growth model of stream network development in a region of uniform lithology and uniform slope is proposed. The principal difference from previous models is that the probability of growth is made dependent on the area contributing runoff to the stream tip. Two versions of the model have been studied in detail and shown to give satisfactory results. A major advantage is that all of the important network variables, including drainage density and outlet density, come naturally out of the simulation processes.

## Introduction

Natural channel networks produced by fluvial erosion exhibit patterns that are related to the geologic structure of the area in which they are incised [1]. Networks formed in regions of negligible geologic controls have a typical dendritic character of the kind illustrated in Fig. 1. In the earliest, largely qualitative, scientific discussions of network patterns, dendritic networks were considered as examples of random or nearly random drainage development. In 1945 Horton [2] showed that quantitative analysis of topological and geometrical properties of such networks disclosed certain regularities that he described in his now-famous laws of drainage composition. Horton's findings were later confirmed and extended by Strahler and his students [3]. In recent theoretical investigations, Shreve [4-6] and Smart [7, 8] showed that assumptions of topological randomness and random distributions of link lengths are sufficient to explain not only the Horton-Strahler empirical laws but also the nature of the observed deviations from those laws.

Simulation techniques have provided an alternative approach to the problem of dendritic drainage development. Most of the simulation models have been based on random walks on a square lattice. In the first such model, proposed by Leopold and Langbein [9], channel networks were generated by downstream growth and coalescence of streams starting from randomly selected sources. The dimensionless parameters (such as bifurcation and length ratios) of the Leopold-Langbein networks

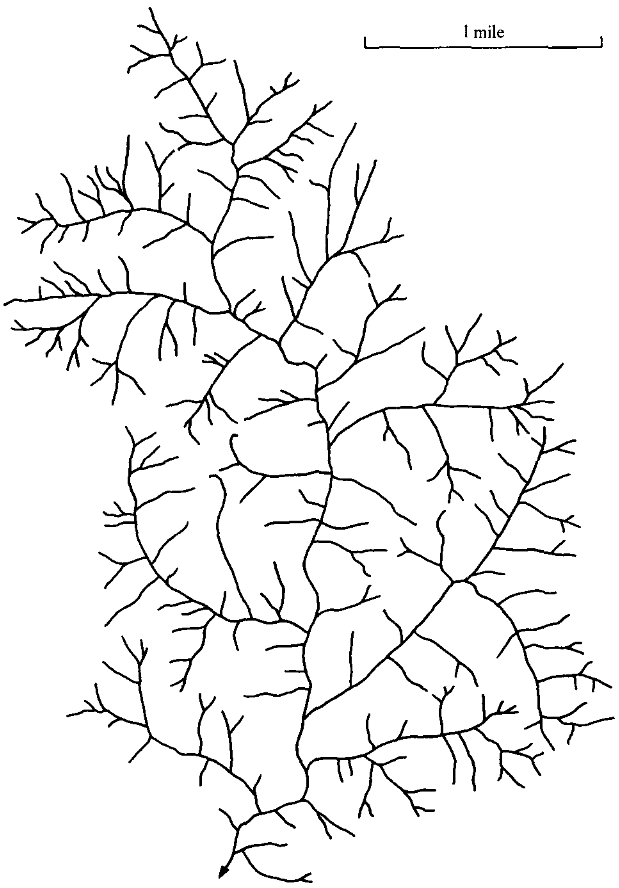


**Figure 1** Dry Creek (Milltown, Ind.) channel network, taken from U. S. Geological Survey map (scale 1:24,000).

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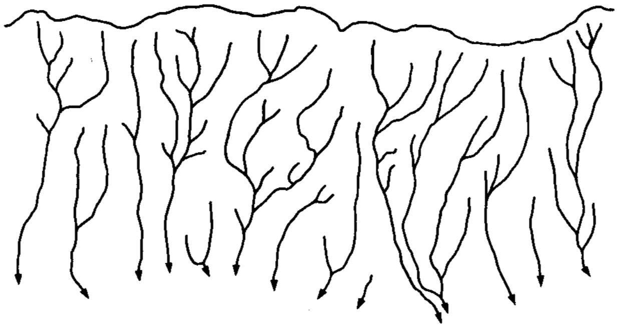
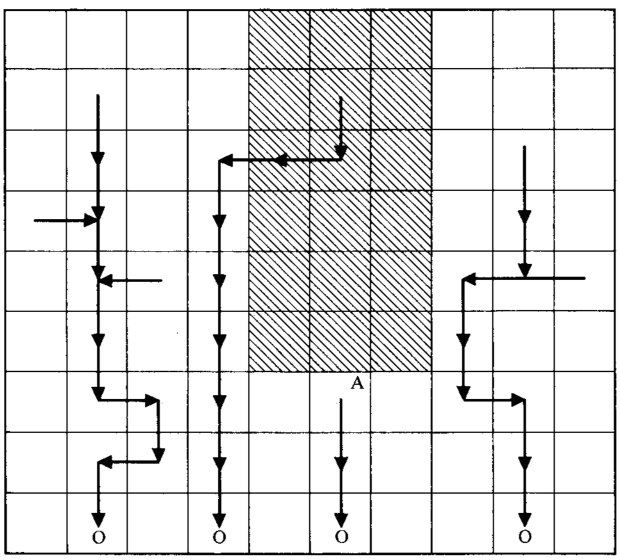


**Figure 2** Drainage network for a homoclinal ridge (Clinch Mountain, Va.)—scarp slope, low dip.

**Figure 3** Headward-growth, random-walk simulation: O—outlets; A—one of several active sites. The potential contributing area for A is indicated by cross-hatching.




were very similar to those of actual systems. Schenck [10] and Smart, Surkan and Considine [11] programmed the model for a digital computer. Later, Howard [12] suggested, programmed and studied several random-walk models in which the networks were generated from randomly selected outlets by headward growth and branching. Other models of the random development of drainage networks have been devised by Scheidegger [13] and by Seginer [14].

One common feature of most of the random-walk models is that the growth probabilities are isotropic. Networks generated under this rule may be expected to correspond to natural networks developed in regions without geologic controls, i.e., in regions of uniform lithology with no pronounced regional slope. Most natural areas, of course, do have some geologic structure, and it would be desirable to generalize the simulation models so that they can be applied to such situations. Smart and Moruzzi [15] made a start in this direction by designing a random-walk, headward-growth model for drainage patterns formed on a homoclinal ridge. The results of their model were tested by comparison with an actual system, Clinch Mountain, Va., which has uniform lithology and structural parameters (dip, relief) that vary along the strike. The drainage network for this region is shown in Fig. 2.

Overall agreement between simulated and actual systems was good, but Smart and Moruzzi noted two specific weaknesses of their model. First, the reduced outlet density, one of the more important geomorphic parameters, had to be specified in the input data, rather than emerge as a natural consequence of the simulation rules. Second, uniform drainage density (channel length per unit area) had to be achieved by making empirical adjustments on two probability ratios whose relation to the actual processes involved in network development was not clear. In the next section we describe a new simulation model intended to remedy these defects.

## Simulation rules

The principal difference between the new and the old models is the replacement of arbitrary probabilities for growth and branching by rules that more closely reflect the actual processes involved. In any drainage system developing by headward growth, one of the most important factors in the rate of growth is the amount of runoff reaching the tip of the stream, and this runoff is in turn largely determined by the upland area contributing to the tip. Accordingly, in the current model, a decision as to whether a given stream segment should be allowed to grow is reached by examining the area immediately upslope. The detailed implementation of such rules is described below.

In its initial form, this model is intended to apply to regions of uniform lithology and uniform regional slope, such as a homoclinal ridge or pediment. As in other random-walk models, an array of squares of horizontal dimension $L$ and vertical dimension $W$ is constructed. The $L$ squares along the bottom row are considered to be possible outlet locations and are identified as *active sites*. The *potential contributing area* for an active site consists of the three columns of squares immediately above and to each side of it; Fig. 3 shows an example for a case in which the active site is not an outlet.

One of the active sites is chosen at random and the V rule (see below) is used to determine whether this site becomes an outlet. The V rule is based on the configuration of streams already in the contributing area; examples are given below. If the site is allowed to become an outlet (as of course it always will for the first active site chosen), a segment of stream is created that runs into the outlet from the square above; this square is then added to the active-site list. The outlet location is removed from the active-site list regardless of the outcome of the V-rule test.

Another site is now chosen at random from the revised active-site list. If the new site is an outlet location, the procedure described above is repeated. If the site is an interior location, the number of streams already entering it is determined and the three neighboring sites (one above, one on each side) are examined to see if they are occupied. A location is removed from the active-site list if it is the junction of two streams or if there is no unoccupied neighboring site; then a new random selection is made. Otherwise, one of the permitted directions of growth (toward an unoccupied site) is chosen at random.

If the *vertical* direction is chosen, the V rule is again used to determine whether growth occurs; if one of the *horizontal* directions is chosen, the H rule is used. In either case, if growth is not allowed the site is removed from the active list; if growth is allowed the newly connected site is added to the active list unless it is in the top row. Next, a new site is chosen at random from the revised active list and the process is repeated until no active site remains.

To obtain an indication of the range and applicability of this model, several versions (i.e., different sets of H and V rules) were programmed and tested. Two particular versions, called simply A and B, are described as follows.

- *Version A*

*V rule* Growth is not allowed if any square in the central column of the potential contributing area is occupied.

*H rule* Growth to the right (left) is not allowed if any square in the right (left) column of the potential contributing area is occupied or if the active site is on the second row from the top.

- *Version B*

*V rule* Same as for version A.

*H rule* Growth is not allowed if any square in the potential contributing area is occupied or if the active site is on the second row from the top.

The general idea behind these rules is that the presence of other streams in the contributing area reduces the runoff available to the active site. The H rules are made more restrictive than the V rules because regional slope will favor channel extension in the "uphill" direction. The major difference between the two versions is illustrated in Fig. 3, where version A would allow no growth at all at active site A, while version B would allow growth to the right only.

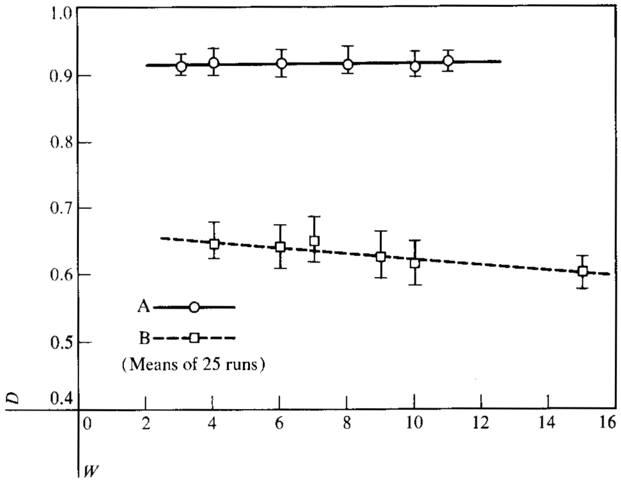



**Figure 4** Mean drainage density vs width for versions A and B.

**Table 1** Dimensionless variables used in stream network analysis.

| *Symbol* | *Definition* | *Description* |
|---|---|---|
| $N_e$ | | Number of exterior links |
| $N_i$ | | Number of interior links |
| $N_o$ | $N_e - N_i$ | Number of outlets |
| $W_r$ | $WD$ | Reduced width |
| $f_B$ | $N_i/N_o$ | Branching factor |
| $\nu_o$ | $N_o/LD$ | Reduced outlet density |
| $\lambda_e$ | $l_eD$ | Reduced mean exterior link length |
| $\lambda_i$ | $l_iD$ | Reduced mean interior link length |
| $\lambda_R$ | $l_e/l_i$ | Mean link-length ratio |
| $\kappa$ | $(N_e + N_i)/WLD^2$ | Reduced link density |
| $\kappa_e$ | $N_e/WLD^2$ | Reduced exterior link density |

## Results of simulation tests

The simulation programs were written in APL\360 and executed on an IBM System/360 Model 91 computer. CPU times for a $10 \times 30$ array were about 18 and 21 seconds for versions B and A, respectively.

Each version was run 25 times for each of six different values of $W$, which were chosen so that the reduced width (width times drainage density) ranged between about 2.5 and 10. For all runs $L$ was set equal to 40, a value considered to be sufficiently large to make end effects negligible and yet sufficiently small to keep computer time to a reasonable amount. With each run data on the numbers and lengths of exterior and interior links were collected and stored to be analyzed as a function of $W$.

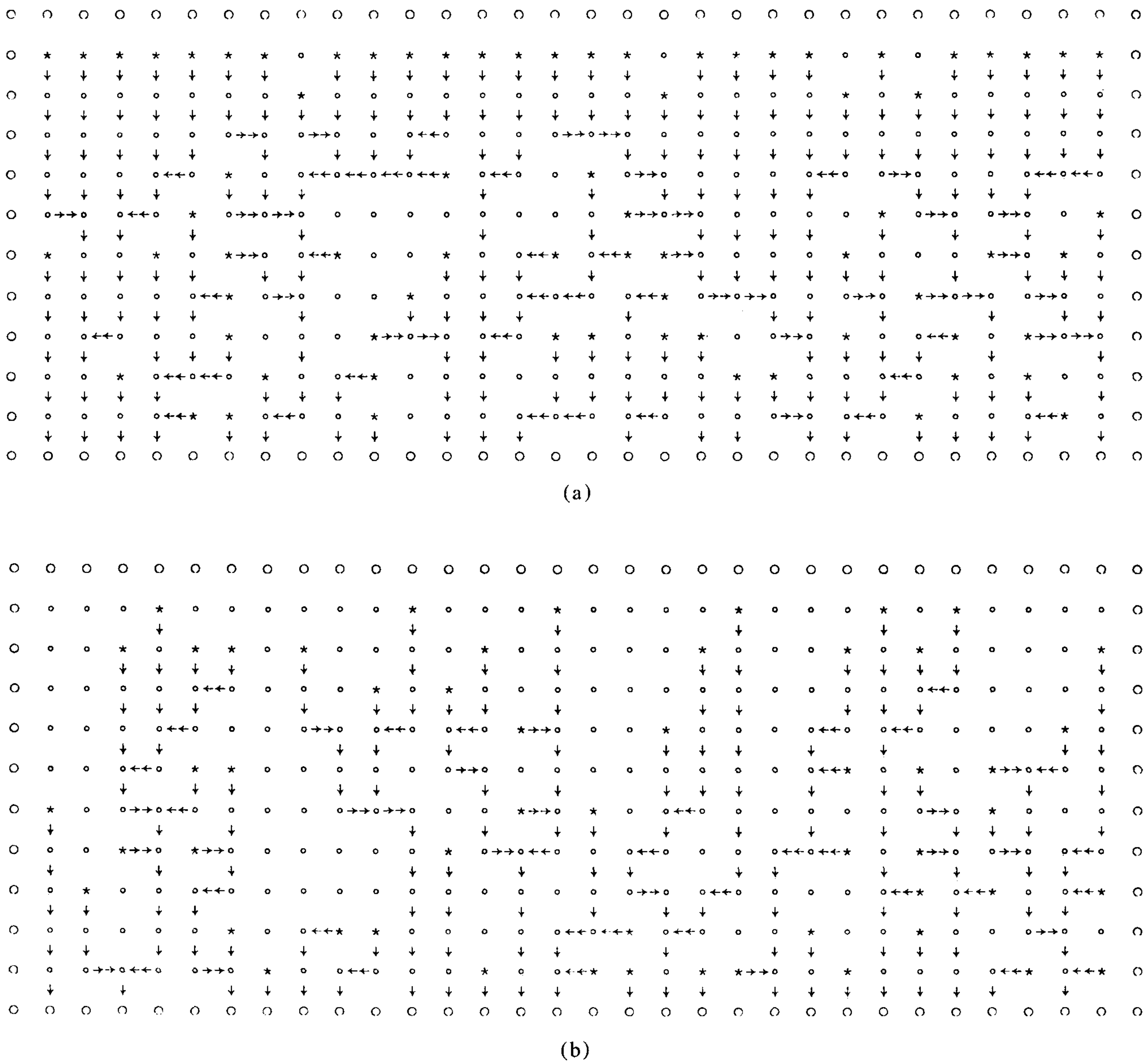



**Figure 5** Typical simulation runs: (a) version A; (b) version B. That the "square" lattice is rectangular, and that there are two arrows in horizontal steps and a single arrow in vertical steps, are artifacts of the APL graphic output.

(An exterior link is a channel segment between a source and the first junction downstream; an interior link is a channel segment between two successive junctions.) As mentioned in our previous paper [15], network data of these types are best reported as dimensionless quantities. The dimensionless variables studied in this investigation are listed in Table 1. The quantity $\kappa$ was introduced into the geomorphological literature by Shreve [5].

One of the most important requirements on simulation models for areas of uniform lithology is that the drainage density $D$ be reasonably uniform and independent of area. In Fig. 4 the mean values of $D$ (for 25 runs) are plotted as a function of $W$ (for constant $L$) for both versions. Seventeen of the 25 observations (68 percent) for each width fall within the range indicated by the brackets. The results based on version A rules show no perceptible dependence of $D$ on $W$, while for version B rules there is an apparently significant decrease in $D$ of about seven percent with increasing $W$ over the range investigated. Version A gives a lesser variability in $D$ than does version B, the coefficients of variation being about 0.02 and 0.055, respectively. It would be interesting to check these values against natural systems, but we have not been able to find any data that are directly comparable. The uniformity of $D$ is harder to characterize quantitatively than is the dependence on area. Computer-generated "maps" of sample runs of both versions are shown in Fig. 5. The reader may judge for himself whether the degree of uniformity is similar to that in natural systems (cf. Fig. 2).

Figure 6 shows the mean values of the branching factor plotted against the mean reduced width. The vertical brackets again indicate the ranges for 17 of 25 observations of $f_B$; the corresponding ranges for $W_r$ are much smaller and are not shown. The relation is so nearly linear that we have made a least-squares fit to an equation of the form

$$f_B = a(W_r - W_0),$$

which is suggested by the fact that in our models no branching occurs unless $W > 2$. Therefore the mean branching factor should go to zero at a value of $W_r \approx 2\bar{D}$. Results are given in Table 2.

The combined dip-slope and scarp-slope data for Clinch Mountain [15] are shown for comparison (here the branching factors are individual values, not means). The occurrence of a non-zero intercept in the natural data suggests the possibility of a critical distance for branching.

Tables 3 and 4 list the mean values for all measured and derived variables. The general behavior of $W$ (or $W_r$) is similar to that produced by our previous simulation program and so is similar to that observed for the Clinch Mountain system. The important point here is that the current model yields results that are as quantitatively satisfactory as those of the previous model while requiring less input data and fewer arbitrary rules.

Some specific features may be noted. The reduced outlet density $\nu_o$ is quite different for the two versions but is essentially independent of $W_r$ in both cases. In version B the mean link-length ratio becomes less than unity for $W_r > 4$, a result that does not agree with most observations. The two reduced link densities behave quite differently; $\kappa_e$ decreases with increasing $W_r$ in both versions, while $\kappa$ shows a slight but significant increase in version A and no change in version B. The reduced exterior link length $\lambda_e$ decreases linearly with $W_r$ both cases.

To give an indication of the inherent variability of each parameter, we have calculated the coefficient of variation for each set of 25 runs and recorded the average of the six values in the last column of both Table 3 and Table 4. The quantities with the lowest coefficients of variation are the drainage density and the number of exterior links.

## Discussion

This model represents a considerable improvement over our previous one. In particular, the drainage development

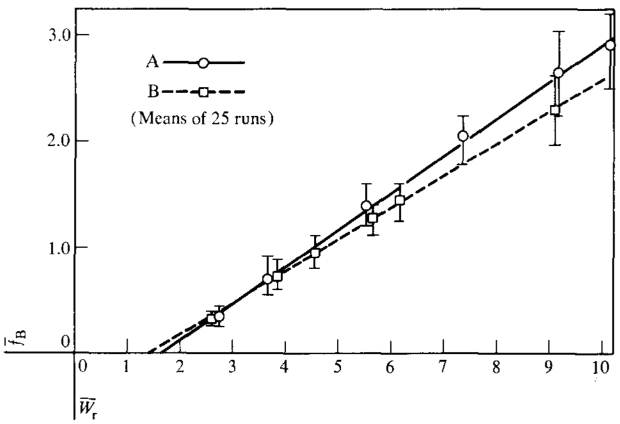



**Figure 6** Mean branching ratio vs mean reduced width for versions A and B. The straight lines represent the least-squares fits with the parameters given in Table 2.

**Table 2** Least-squares fit to branching-factor data.

| *Data* | *a* | $W_0$ | $2\bar{D}$ | *Correlation coefficient* |
|---|---|---|---|---|
| Version A | 0.351 | 1.65 | 1.83 | 0.999 |
| Version B | 0.302 | 1.45 | 1.26 | 0.998 |
| Clinch Mountain | 0.434 | 2.14 | | 0.925 |

terminates automatically, much as in actual systems. The only input data required are the width and length of the map; all of the network variables (including $D$ and $N_o$, which had to be specified as input in the previous model) come naturally out of the simulation process.

All of the essential properties of the model are contained in the H and V rules. This feature provides a great deal of flexibility, since different versions are created simply by changing the criteria for growth. Thus, to simulate drainage development in an area underlain by a permeable, erosion-resistant rock such as sandstone, one would require that a large fraction of the contributing area be unoccupied before growth can occur; such a choice automatically leads to an appropriately low drainage density. Another attractive feature of our model is that it is easy to change from one version of the program to another, since only the statements relating to the H and V rules (four, in our case) need be altered.

The two versions used to illustrate the properties and range of the model were chosen for their simplicity. We believe that more realistic versions are certainly possible, but that the details will depend rather sensitively on the type of area being simulated. We offer here two general suggestions for improvement to be incorporated in any more sophisticated model. First, in the current versions, if horizontal (vertical) growth is not allowed,

**Table 3** Mean values of network variables for version A.

| Variable | W: 3 | 4 | 6 | 8 | 10 | 11 | Coefficient of variation |
|---|---|---|---|---|---|---|---|
| $D$ | 0.913 | 0.919 | 0.918 | 0.917 | 0.913 | 0.921 | 0.022 |
| $W_r$ | 2.74 | 3.67 | 5.51 | 7.34 | 9.13 | 10.13 | |
| $N_e$ | 39.6 | 48.4 | 66.2 | 84.4 | 101.6 | 112.8 | 0.046 |
| $N_i$ | 9.92 | 19.9 | 38.3 | 56.5 | 73.6 | 83.8 | 0.114 |
| $N_o$ | 29.6 | 28.6 | 27.9 | 27.9 | 28.1 | 29.0 | 0.092 |
| $f_B$ | 0.34 | 0.71 | 1.39 | 2.05 | 2.65 | 2.92 | 0.164 |
| $\nu_o$ | 0.811 | 0.777 | 0.761 | 0.761 | 0.769 | 0.787 | 0.093 |
| $\lambda_e$ | 2.30 | 2.26 | 2.15 | 2.07 | 2.01 | 2.01 | 0.078 |
| $\lambda_i$ | 0.91 | 1.31 | 1.57 | 1.69 | 1.77 | 1.74 | 0.084 |
| $\lambda_R$ | 2.52 | 1.73 | 1.38 | 1.22 | 1.14 | 1.17 | 0.093 |
| $\kappa$ | 0.495 | 0.507 | 0.518 | 0.524 | 0.526 | 0.527 | 0.072 |
| $\kappa_e$ | 0.396 | 0.360 | 0.328 | 0.314 | 0.305 | 0.302 | 0.062 |

**Table 4** Mean values of network variables for version B.

| Variable | W: 4 | 6 | 7 | 9 | 10 | 15 | Coefficient of variation |
|---|---|---|---|---|---|---|---|
| $D$ | 0.645 | 0.641 | 0.651 | 0.626 | 0.616 | 0.604 | 0.054 |
| $W_r$ | 2.58 | 3.58 | 4.56 | 5.63 | 6.16 | 9.06 | |
| $N_e$ | 43.2 | 54.6 | 60.5 | 71.4 | 75.6 | 103.6 | 0.067 |
| $N_i$ | 10.5 | 22.8 | 29.2 | 31.5 | 44.4 | 71.9 | 0.143 |
| $N_o$ | 32.8 | 31.8 | 31.3 | 31.5 | 31.2 | 31.8 | 0.072 |
| $f_B$ | 0.32 | 0.73 | 0.94 | 1.27 | 1.44 | 2.28 | 0.183 |
| $\nu_o$ | 1.27 | 1.24 | 1.20 | 1.26 | 1.27 | 1.32 | 0.086 |
| $\lambda_e$ | 1.31 | 1.28 | 1.31 | 1.22 | 1.17 | 1.15 | 0.098 |
| $\lambda_i$ | 0.98 | 1.29 | 1.37 | 1.37 | 1.43 | 1.40 | 0.113 |
| $\lambda_R$ | 1.37 | 1.00 | 0.96 | 0.90 | 0.83 | 0.82 | 0.128 |
| $\kappa$ | 0.809 | 0.785 | 0.759 | 0.794 | 0.793 | 0.805 | 0.084 |
| $\kappa_e$ | 0.651 | 0.555 | 0.512 | 0.510 | 0.500 | 0.476 | 0.083 |

the site is removed from the active list, although vertical (horizontal) growth might still be permissible. This minor inconsistency can be remedied by using two active lists, one for horizontal growth and one for vertical growth. Second, it would probably be more realistic to replace the yes-no decisions on growth in the current versions with a random choice in which the probability of growth is a function of the fraction of potential contributing area that is unoccupied and unshadowed. Intuitively, the dependence of growth probability on area could be represented satisfactorily by a linear increase from a positive threshold, followed by an asymptotic approach to unity. This change would be particularly important if one wanted to simulate the time dependence of drainage development.

This model could also in principle be generalized to regions with complex geologic controls, simply by making the H and V rules appropriate functions of location. Obviously, any attempt to make a detailed simulation of a very complex region may result in unacceptable requirements on computer memory and execution time.

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