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Asymptotic Expansion for Small Magnetic Fields of Acoustoelectric Attenuation in Nondegenerate Semiconductors

Abstract: The semiclassical analysis of acoustoelectric effects involves an infinite sum $S(c,x) = ic \exp(-x) \sum_{n=-\infty}^{+\infty} (n+ic)^{-1} I_n(x)$, in which both arguments c and x depend on the magnetic field B. Recently Lebwohl, Carlson, and Mosekilde have found an integral representation for this sum, through which now we identify $S(c,x)$ as a generalized hypergeometric function. Moreover we derive an asymptotic series for $S(c,x)$ in the limit of small B, whose coefficients, in a parameter z, involve the iterated integrals of the complementary error function.

Introduction

In recent years Spector [1,2] has considered sound-wave propagation in nondegenerate semiconductors and provided a semiclassical discussion of the acoustoelectric effect, while Route and Kino [3] have generalized Spector's treatment to drifting electron distributions and compared their analytical result with experiments in InSb. In this work the expressions for conductivity and absorption contain the infinite sum

$$S(c,x) = ic \exp(-x) \sum_{n=-\infty}^{+\infty} (n+ic)^{-1} I_n(x), \tag{1}$$

which is sometimes hard to evaluate numerically. Here $I_n(x)$ is a modified Bessel function and

$$c = (1+i\nu\tau)/\mu B,$$
$$x = (kl/\mu B)^2/2,$$
$$\nu = \omega - kw, \tag{2}$$

where ω is the acoustic frequency, k is the acoustic wave number, B is the magnetic field strength, w is the electron drift velocity, τ is the electron relaxation time, l is the electron mean free path, and μ is the zero-field electron mobility.

More immediately Lebwohl, Carlson, and Mosekilde [4] have rewritten the sum (1) as an integral

$$S(c,x) = [c/\sinh(c\pi)] \times \int_0^\pi \exp(x\cos\theta - x)\cosh(c\pi - c\theta)\,d\theta, \tag{3}$$

have approximated this integral for small B, and have recovered an expression for $B=0$ obtained previously by Route and Kino [3]. They have also pointed out the chief problem in this analysis: the integral for $S(c,x)$ as a function of B has an essential singularity at the point $B=0$. In this communication we derive a systematic expansion as $B \to 0+$, but first we obtain some new representations for the integral (3). These should afford some additional insight into the functional form of $S(c,x)$.

Let us regard c as a fixed parameter and consider $S(c,x)$ as a function of x. If also we put $\phi = \pi - \theta$ and define

$$T(c,x) = \exp(x)S(c,x) = [c/\sinh(c\pi)] \int_0^\pi \exp(-x\cos\phi)\cosh(c\phi)\,d\phi, \tag{4}$$

then the resulting integral for $T(c,x)$ suggests a representation of $I_{\pm ic}(x)$ [5,Eq.(9.6.20)]. Indeed through substitution from (4) and integration by parts we find that

$$x^2 d^2T/dx^2 + x\,dT/dx + (c^2 - x^2)T = c^2 \exp(x). \tag{5}$$

Also, by examination of (4), we note that $T(c,x)$ is analytic for $|x| < \infty$ and that

$$S(c,0) = T(c,0) = 1. \tag{6}$$

This ordinary differential equation is solvable through variation of parameters, since the corresponding homogeneous equation is satisfied by

$$T_h(c,x) = C_+ I_{+ic}(x) + C_- I_{-ic}(x). \tag{7}$$

Moreover, any function (7) with nontrivial constants $C_\pm$ has a branch point at the origin, whence the problem (5)-(6) has a unique solution $T(c,x)$ with a Taylor expansion at the origin.

Also, from (4) and (5), we find that

$$x^2 d^2S/dx^2 + (2x^2 + x)dS/dx + (x + c^2)S = c^2 \qquad (8)$$

and by the preceding remarks we may assume that

$$S(c,x) = \sum_{n=0}^{\infty} s_n(c)x^n \qquad \text{with } s_0(c) = 1. \qquad (9)$$

By substitution we obtain, for $n > 0$, the recursion formula

$$(n^2 + c^2)s_n(c) = (n - \tfrac{1}{2})[-2s_{n-1}(c)] \qquad (10)$$

and thus, for $S(c,x)$, the convergent series

$$S(c,x) = {}_2F_2(\tfrac{1}{2}, 1;\ 1 + ic,\ 1 - ic; -2x)$$
$$= \sum_{n=0}^{\infty} \frac{(-2x)^n(n - \frac{1}{2})\cdots(1 - \frac{1}{2})}{(n + ic)\cdots(1 + ic)(n - ic)\cdots(1 - ic)}. \qquad (11)$$

In other words, $S(c,x)$ can be expressed [6, pp.373–384] as a generalized hypergeometric function, ${}_2F_2$, and thus as a Meijer G-function, $G_{2,3}{}^{1,2}$. A search among special identities for such functions suggests that we should not expect a simpler form for $S(c,x)$.

Expansion

Both arguments in $S(c,x)$ are functions of B. Hence, to achieve a convenient form for the desired expansion, we introduce the new variables

$$t = kl/\mu B,$$
$$u = \sin(\theta/2),$$
$$z = (1 + i\nu\tau)/kl, \qquad (12)$$

and we decompose the integral (3):

$$S(c,x) = [c/\sinh(c\pi)] \times [\exp(c\pi)R(z,t) + \exp(-c\pi)R(-z,t)],$$
$$R(z,t) = \int_0^1 \exp(-t^2u^2 - 2ztu)P(zt,u)du,$$
$$P(zt,u) = (1 - u^2)^{-\frac{1}{2}} \exp[2ztu - 2zt \sin^{-1}(u)]. \qquad (13)$$

The expansion of $\sin^{-1}(u) - u$ in powers of u has leading term $u^3/6$, whence the expansion of $P(zt,u)$ in powers of u has general form

$$P(zt,u) = \sum_{m=0}^{\infty} p_m(zt)u^m \qquad (14)$$

with p_m a polynomial of degree $\leq m/3$. However, for any complex z, we note that

$$\int_0^{\infty} \exp(-t^2u^2 - 2ztu)u^n du$$
$$= t^{-n-1}(-2)^{-n}(\partial/\partial z)^n \int_0^{\infty} \exp(-v^2 - 2zv)dv$$
$$= \tfrac{1}{2}\pi^{\frac{1}{2}}t^{-n-1}(-2)^{-n}(\partial/\partial z)^n \exp(z^2) \operatorname{erfc}(z)$$
$$= \tfrac{1}{2}\pi^{\frac{1}{2}}n!t^{-n-1} \exp(z^2)\mathbf{i}^n \operatorname{erfc}(z), \qquad (15)$$

where $\mathbf{i}^n \operatorname{erfc}(z)$ is the nth iterated integral of the complementary error function [5, Eq.(7.2.9)]. If we substitute (14) into $R(z,t)$, extend the integration to $+\infty$, and integrate term by term, then we obtain a formal expansion

$$R(z,t) \sim \tfrac{1}{2}\pi^{\frac{1}{2}} \sum_{m=0}^{\infty} m!p_m(zt)t^{-m-1} \exp(z^2)\mathbf{i}^m \operatorname{erfc}(z) \qquad (16)$$

as $t \to +\infty$. Now (16) can be rearranged as a series in t^{-1} with coefficients involving z, whence $S(c,x)$ can be expressed as a series in B by (12) and (13). We need only show that (16) is an asymptotic series for large t.

These manipulations might perhaps be justified through some general theorem of asymptotic analysis (e.g., [7]), but they can quickly be validated through a few direct estimates of remainder terms. Indeed, for any real a and b with $0 < a < b < 1$, we note

$$\int_b^1 \exp(-t^2u^2 - 2ztu)P(zt,u)du = o[\exp(-a^2t^2)],$$
$$\int_b^{\infty} \exp(-t^2u^2 - 2ztu)u^n du = o[\exp(-a^2t^2)], \qquad (17)$$

as $t \to +\infty$. On the interval $[0,b]$, furthermore, (14) converges absolutely and uniformly, so that

$$|P(zt,u) - \sum_{m=0}^{n-1} p_m(zt)u^m| \leq K(b)\ |zt|^{n/3}\ u^n. \qquad (18)$$

Thus, by (17) and (18),

$$|R(z,t) - \sum_{m=0}^{n-1} p_m(zt) \int_0^{\infty} \exp(-t^2u^2 - 2ztu)u^m du|$$
$$\leq o[\exp(-a^2t^2)]$$
$$+ K(b)|zt|^{n/3} \int_0^b \exp(-t^2u^2 + 2|z|tu)u^n du$$
$$\leq o[\exp(-a^2t^2)] + O(t^{-1-2n/3}) \qquad \text{as } t \to +\infty. \qquad (19)$$

We have now verified the expansion (16) and need only compute the first few terms. However, from (13)

$$P(zt,u) = 1 + (1/2)u^2 - (zt/3)u^3$$
$$+ (3/8)u^4 - (19\,zt/60)u^5$$
$$+ [(z^2t^2/30) + (5/16)]u^6 + O(ztu^7 + z^2t^2u^8) \qquad (20)$$

as $u \to 0$; and thus, from (16),

$$R(z,t) = \tfrac{1}{2}\pi^{\frac{1}{2}} \exp(z^2)[t^{-1}\mathbf{i}^0 + t^{-3}(\mathbf{i}^2 - 2z\mathbf{i}^3)$$
$$+ t^{-5}(9\mathbf{i}^4 - 38z\mathbf{i}^5 + 24z^2\mathbf{i}^6) + O(t^{-7})] \operatorname{erfc}(z) \qquad (21)$$

as $t \to +\infty$. By the detailed form of the expansion for $P(zt,u)$, only odd powers of t can occur in (21). Finally, from (13),

$$2\sinh(\pi zt)S(c,x)/\pi^{\frac{1}{2}} z\exp(z^2)$$

$$\sim \exp(\pi zt)[\mathbf{i}^0 + t^{-2}(\mathbf{i}^2 - 2z\mathbf{i}^3)$$

$$+ t^{-4}(9\mathbf{i}^4 - 38z\mathbf{i}^5 + 24z^2\mathbf{i}^6) + \cdots]\operatorname{erfc}(z)$$

$$+ \exp(-\pi zt)[\mathbf{i}^0 + t^{-2}(\mathbf{i}^2 + 2z\mathbf{i}^3)$$

$$+ t^{-4}(9\mathbf{i}^4 + 38z\mathbf{i}^5 + 24z^2\mathbf{i}^6) + \cdots]\operatorname{erfc}(-z) \qquad (22)$$

as $t \to +\infty$. We may retain exponentially small terms if we interpret (22) as a multiple asymptotic expansion in the sense of Shere [8]. If we substitute $t = kl/\mu B$ from (12), then we get the desired expansion for small B.

Through the recursion formulas [5, Eq. (7.2.5)]

$$n\mathbf{i}^n \operatorname{erfc}(z) + z\mathbf{i}^{n-1}\operatorname{erfc}(z) - \tfrac{1}{2}\mathbf{i}^{n-2}\operatorname{erfc}(z) = 0, \qquad (23)$$

the functions $\mathbf{i}^m \operatorname{erfc}(z)$ in these series can all be computed numerically via an algorithm of Gautschi [9, 10] or related simply to the basic pair

$$\mathbf{i}^{-1}\operatorname{erfc}(z) = 2\pi^{\frac{1}{2}}\exp(-z^2),$$

$$\mathbf{i}^0 \operatorname{erfc}(z) = \operatorname{erfc}(z). \qquad (24)$$

The leading term of (22) is the result of Lebwohl, Carlson, and Mosekilde [4], but it appears in our analysis as part of a complete expansion. We hope that (22) will facilitate the understanding of acoustoelectric phenomena in the region of small B.

References

1. H. N. Spector, "Magnetoacoustic effects in nondegenerate semiconductors," *Phys. Rev.* **125,** 1880 (1962).
2. H. N. Spector, "Magnetic field dependence of the amplification of sound by conduction electrons," *Phys. Rev.* **131,** 2512 (1963).
3. R. K. Route and G. S. Kino, "Acoustoelectric amplification in InSb," *IBM J. Res. Develop.* **13,** 507 (1969).
4. P. A. Lebwohl, D. G. Carlson, and E. Mosekilde, "On the acoustoelectric gain in magnetic fields," *J. Appl. Phys.* **43,** 1284 (1972).
5. M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions*, U. S. Government Printing Office, Washington, D.C., 1964.
6. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, Vol. 1, McGraw Hill Book Co. Inc., New York, 1954.
7. D. S. Jones, "Asymptotic behavior of integrals," *SIAM Review* **14,** 286 (1972).
8. K. D. Shere, "On multiple asymptotic expansions," *SIAM J. Math. Anal.* **3,** 272 (1972).
9. W. Gautschi, "Recursive computation of the repeated integrals of the error function," *Math. Comp.* **15,** 227 (1961).
10. W. Gautschi, "Efficient computation of the complex error function," *SIAM J. Numer. Anal.* **7,** 187 (1970).

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