The four-parameter kappa distribution

by J. R. M. Hosking

Many common probability distributions, including some that have attracted recent interest for flood-frequency analysis, may be regarded as special cases of a four-parameter distribution that generalizes the three-parameter kappa distribution of P. W. Mielke. This four-parameter kappa distribution can be fitted to experimental data or used as a source of artificial data in simulation studies. This paper describes some of the properties of the four-parameter kappa distribution, and gives an example in which it is applied to modeling the distribution of annual maximum precipitation data.

Introduction

The transformation

$$X = \begin{cases} \xi + \alpha(1 - e^{-kY})/k, & \text{if } k \neq 0, \\ \xi + \alpha Y, & \text{if } k = 0, \end{cases} \tag{1}$$

where X and Y are real-valued random variables and ξ, α, and k are real-valued parameters, underlies several distributions recently used in flood-frequency analysis. If Y has an exponential distribution, with cumulative distribution function

$$F(y) = 1 - e^{-y}, \qquad y \geq 0, \tag{2}$$

then X has a three-parameter generalized Pareto distribution, used in [1]. A two-parameter form of the distribution, lacking the location parameter ξ, has also been used in [2] and [3]. If Y has a Gumbel (extreme-value type I) distribution, with cumulative distribution function

$$F(y) = \exp(-e^{-y}), \tag{3}$$

then X has a generalized extreme-value (GEV) distribution, used in [4–6]. If Y has a logistic distribution, with cumulative distribution function

$$F(y) = \frac{1}{1 + e^{-y}}, \tag{4}$$

then X has what, following [7], we call a generalized logistic distribution. It is a convenient reparameterization of the log–logistic distribution used in [8].

The functions (2)–(4) may be regarded as special cases for $h = 1$, $h = 0$, and $h = -1$, respectively, of the cumulative distribution function

$$F(y) = \begin{cases} (1 - he^{-y})^{1/h}, & \text{for } y \geq \log h, \quad \text{if } h \neq 0, \\ \exp(-e^{-y}), & \text{if } h = 0. \end{cases} \tag{5}$$

Thus, we can define a four-parameter distribution that includes the generalized Pareto, GEV, and generalized logistic distributions as special cases. It is the distribution of a random variable X that is obtained by applying the transformation (1) to a random variable Y distributed as in (5). We call this the four-parameter kappa distribution, because, as explained below, it can be regarded as a generalization of the three-parameter kappa distribution used in [9]. The cumulative distribution function of the four-parameter kappa distribution is

$$F(x) = \begin{cases} \{1 - h[1 - k(x - \xi)/\alpha]^{1/k}\}^{1/h}, & \text{if } k \neq 0, h \neq 0, \\ \exp\{-[1 - k(x - \xi)/\alpha]^{1/k}\}, & \text{if } k \neq 0, h = 0, \\ \{1 - h \exp[-(x - \xi)/\alpha]\}^{1/h}, & \text{if } k = 0, h \neq 0, \\ \exp\{-\exp[-(x - \xi)/\alpha]\}, & \text{if } k = 0, h = 0. \end{cases} \tag{6}$$

The four-parameter kappa distribution has several useful features. As a generalization of the generalized logistic, generalized extreme-value, and generalized Pareto distributions, it is a candidate for being fitted to data when these three-parameter distributions give an inadequate fit, or when the experimenter does not want to be committed to the use of a particular three-parameter distribution. It can also be used in robustness studies, to check the extent to which a statistical procedure remains valid when its distributional assumptions are not valid. In this spirit, Hosking and Wallis [10] used the four-parameter kappa distribution to generate artificial data for assessing the goodness of fit of different distributions.

In a particular application, the h parameter of the four-parameter kappa distribution can be given a physical interpretation, as follows. Suppose that environmental events (e.g., storms and floods) with magnitudes $Z_1, Z_2, \cdots, Z_N$ occur in a year, with the number of events N itself being a random variable. This is a common model specification in "peak-over-threshold" or "partial-duration series" modeling of environmental events [11, 12]. Suppose that each event magnitude Z_i has a generalized Pareto distribution, with cumulative distribution function

$$P[Z_i < z] = F_Z(z) = 1 - [1 - k(z - \xi^*)/\alpha^*]^{1/k}.$$

(We use ξ^* and α^* here to differentiate these parameters from the ξ and α of the four-parameter kappa distribution.) Suppose also that the magnitudes of different events are statistically independent and that N, the number of events in a year, has a binomial distribution, with parameters n and p, defined by

$$P[N = j] = \binom{n}{j} p^j (1 - p)^{n-j}, j = 0, 1, \cdots, n, \qquad 0 < p < 1.$$

Then, the year's largest event has a distribution with cumulative distribution function given by

$$\begin{aligned} P[\max(Z_1, \cdots, Z_N) \le z] &= P[Z_i \le z, i = 1, \cdots, N] \\ &= \sum_{j=0}^{n} P[N = j] P[Z_i \le z, i = 1, \cdots, j] \\ &= \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} [F_Z(z)]^j \\ &= \{1 - p[1 - F_Z(z)]\}^n \\ &= \{1 - p[1 - k(z - \xi^*)/\alpha^*]^{1/k}\}^n. \end{aligned}$$

This can be written as the cumulative distribution function of a four-parameter kappa distribution with $h = 1/n$ and a k parameter equal to that of the generalized Pareto distribution of the Z_i. Thus $1/h$ can be interpreted as the maximum potential number of events in a year. In the limiting case of $n \to \infty$ while np remains constant, the binomial distribution becomes a Poisson distribution, and the distribution of the annual maximum is a four-parameter kappa distribution with $h = 0$, i.e., a generalized extreme-value distribution, as noted in [3].

Special cases

Several established distributions are special cases of the four-parameter kappa distribution. As noted above, $h = 1$ yields a generalized Pareto distribution, $h = 0$ a GEV distribution, and $h = -1$ a generalized logistic distribution. An exponential distribution arises when $h = 1$ and $k = 0$, a Gumbel distribution when $h = 0$ and $k = 0$, a logistic distribution when $h = -1$ and $k = 0$, and a uniform distribution when $h = 1$ and $k = 1$. When $h = 0$ and $k = 1$, the four-parameter kappa distribution is a reverse exponential distribution; i.e., $1 - F(-x)$ is the cumulative distribution function of an exponential distribution.

Mielke [9] defined the three-parameter kappa distribution by its cumulative distribution function

$$F(x) = (x/b)^\theta [a + (x/b)^{a\theta}]^{-1/a}, \quad x \ge 0, \quad a, b, \theta > 0. \tag{7}$$

This is equivalent to (6) when $\xi = b$, $\alpha = b/a\theta$, $k = -1/a\theta$, and $h = -a$. The cumulative distribution function (7) includes those cases of (6) in which $h < 0$, $k < 0$, and ξ is chosen so that the lower bound of the distribution is zero. Thus, the four-parameter kappa distribution may be regarded as a reparameterization of the three-parameter kappa distribution with an additional location parameter.

The four-parameter kappa distribution is also related to two of the families of distributions defined by Burr [13]. When $h < 0$ and $k < 0$, (6) is a Burr type III distribution; when $h < 0$ and $k > 0$, (6) is a reverse Burr type XII distribution. These distributions have several applications: as failure-time distributions in reliability studies [14, 15], as distributions of actuarial data [16], and as approximations to distributions of test statistics [17]. A general discussion of Burr distributions is given in [18].

Properties of the four-parameter kappa distribution

It is convenient to write the cumulative distribution function of the four-parameter kappa distribution simply as

$$F(x) = \{1 - h[1 - k(x - \xi)/\alpha]^{1/k}\}^{1/h}, \tag{8}$$

which is to be understood as including the cases $h = 0$ and $k = 0$, since the other forms of $F(x)$ in (6) are the limiting forms of (8) as $k \to 0$ or $h \to 0$. With the same understanding, the distribution has probability density function

$$f(x) = \alpha^{-1}[1 - k(x - \xi)/\alpha]^{(1/k)-1}[F(x)]^{1-h}$$

and quantile function (inverse cumulative distribution function)

$$x(F) = \xi + \frac{\alpha}{k}\left[1 - \left(\frac{1 - F^h}{h}\right)^k\right].$$

Of the four parameters, ξ is a location parameter, α is a scale parameter, and k and h are shape parameters. Apart from the restriction $\alpha > 0$, all parameter values yield valid distribution functions. Noting that $(1 - F^h)/h$ has limits 0 as $F \nearrow 1$, $+\infty$ as $F \searrow 0$ with $h < 0$, and h^{-1} as $F \searrow 0$ with $h > 0$, we can calculate the bounds of the distribution. They are as follows:

$$\begin{array}{rl}
\xi + \alpha(1 - h^{-k})/k \le x \le \xi + \alpha/k & \text{if } h > 0, k > 0; \\
\xi + \alpha \log h \le x < \infty & \text{if } h > 0, k = 0; \\
\xi + \alpha(1 - h^{-k})/k \le x < \infty & \text{if } h > 0, k < 0; \\
-\infty < x \le \xi + \alpha/k & \text{if } h \le 0, k > 0; \\
-\infty < x < \infty & \text{if } h \le 0, k = 0; \\
\xi + \alpha/k \le x < \infty & \text{if } h \le 0, k < 0.
\end{array}$$

The probability density function has at most one extremum. More precisely: $f(x)$ has a single maximum if $h < 0$ and $1/h < k < 1$, or if $0 \le h < 1$ and $k < 1$; $f(x)$ has a single minimum if $h > 1$ and $k > 1$; otherwise $f(x)$ has no extremum. Thus the graph of $f(x)$ has essentially four possible shapes; these are illustrated in **Figure 1.**

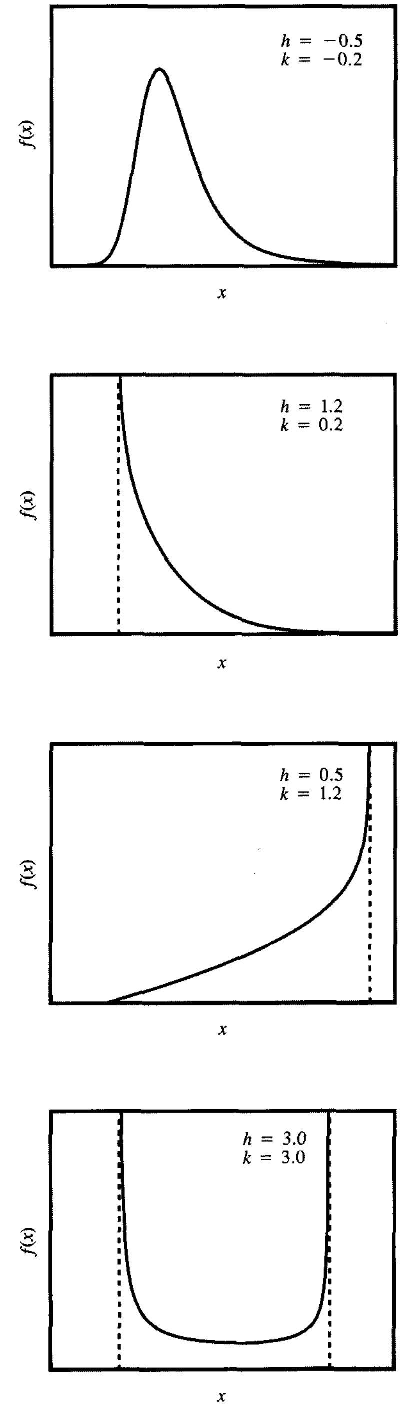

Figure 1

Examples of probability density functions of four-parameter kappa distributions. Dotted lines indicate bounds of the distribution at which the density becomes infinite.

Moments

The rth noncentral moment of a probability distribution is

$$E(X^r) = \int_0^1 [x(F)]^r \, dF.$$

The rth moment (r is a positive integer) of the four-parameter kappa distribution exists (i.e., this integral is finite) under the following circumstances: for all r, if $h \geq 0$ and $k \geq 0$; for $r < -1/hk$, if $h < 0$ and $k \geq 0$; for $r < -1/k$, if $k < 0$. When $h \neq 0$ and $k \neq 0$, moments of integral order may be obtained from the following expression:

$$E[1 - k(X - \xi)/\alpha]^r = \int_0^1 [h^{-1}(1 - F^h)]^{rk}\, dF. \tag{9}$$

When $h > 0$, we substitute $u = F^h$, and the integral becomes

$$h^{-(1+rk)} \int_0^1 (1 - u)^{rk} u^{(1/h)-1}\, du = h^{-(1+rk)} \frac{\Gamma(1 + rk)\Gamma(1/h)}{\Gamma(1 + rk + 1/h)}$$

([19], Equation 3.191.3), where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ is the gamma function. When $h < 0$, we substitute $u = F^{-h} - 1$, and the integral becomes

$$(-h)^{-(1+rk)} \int_0^\infty u^{rk}(1 + u)^{(1/h)-1}\, du$$

$$= (-h)^{-(1+rk)} \frac{\Gamma(1 + rk)\Gamma(-rk - 1/h)}{\Gamma(1 - 1/h)}$$

([19], Equation 3.194.3). Thus,

$$E[1 - k(X - \xi)/\alpha]^r = \begin{cases} h^{-(1+rk)} \dfrac{\Gamma(1 + rk)\Gamma(1/h)}{\Gamma(1 + rk + 1/h)}, & h > 0, \\ (-h)^{-(1+rk)} \dfrac{\Gamma(1 + rk)\Gamma(-rk - 1/h)}{\Gamma(1 - 1/h)}, & h < 0. \end{cases}$$

Even when the first four moments of the distribution are known, it is not always possible to determine uniquely the parameters of the distribution. In general, the four-parameter kappa distribution can have the same set of first four moments for two different sets of parameters. This is because some skewness–kurtosis combinations correspond to two distinct (h, k) pairs. The form of the cumulative distribution function (6) gives no indication why this should be the case, but that it is so may be seen from the moment–ratio diagrams of Burr distributions given in [18]. The same phenomenon occurs with the L-moments of the four-parameter kappa distribution, discussed in the following section.

Probability-weighted moments and *L*-moments

Probability-weighted moments (PWMs) of a random variable X with cumulative distribution function F are defined [20] to be the quantities

$$M_{p,r,s} = E\{X^p[F(X)]^r[1 - F(X)]^s\}.$$

Particularly useful special cases are the PWMs

$$\beta_r = M_{1,r,0} = E\{X[F(X)]^r\}, \qquad r = 0, 1, \cdots. \tag{10}$$

L-moments, defined in [7, 21], are linear combinations of probability-weighted moments:

$$\lambda_1 = \beta_0,$$

$$\lambda_2 = 2\beta_1 - \beta_0,$$

$$\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0,$$

$$\lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0. \tag{11}$$

In general,

$$\lambda_{r+1} = \sum_{j=0}^{r} p^*_{r,j}\beta_j, \qquad r = 0, 1, \cdots,$$

where

$$p^*_{r,j} = (-1)^{r-j}\binom{r}{j}\binom{r+j}{j} = \frac{(-1)^{r-j}(r + j)!}{(j!)^2(r - j)!}.$$

"L-moment ratios" τ_r, where $r = 3, 4, \cdots$, are defined by $\tau_r = \lambda_r/\lambda_2$.

L-moments and L-moment ratios are more convenient than probability-weighted moments, because they are more easily interpretable as measures of distributional shape. In particular, λ_1 is the mean of the distribution, a measure of location; λ_2 is a measure of scale; and τ_3 and τ_4 are measures of skewness and kurtosis, respectively.

The foregoing quantities are defined for a probability distribution, but in practice they must often be estimated from a finite sample. Let $x_1 \leq x_2 \leq \cdots \leq x_n$ be the ordered sample. Let

$$b_r = n^{-1} \sum_{i=1}^{n} \frac{(i - 1)(i - 2)\cdots(i - r)}{(n - 1)(n - 2)\cdots(n - r)} x_i, \qquad r = 0, 1, \cdots, n - 1,$$

and

$$\ell_{r+1} = \sum_{j=0}^{r} p^*_{r,j} b_j, \qquad r = 0, 1, \cdots, n - 1.$$

Then b_r and ℓ_r are unbiased estimators of β_r and λ_r, respectively. The estimator $t_r = \ell_r/\ell_2$ of τ_r is consistent but not unbiased. The quantities ℓ_1, ℓ_2, t_3, and t_4 are useful summary statistics of a sample of data. They can be used to identify the distribution from which a sample was drawn ([21], Section 3.5). They can also be used to estimate parameters when fitting a distribution to a sample, by equating the sample and population L-moments ([21], Section 4.1). L-moments are in several respects superior

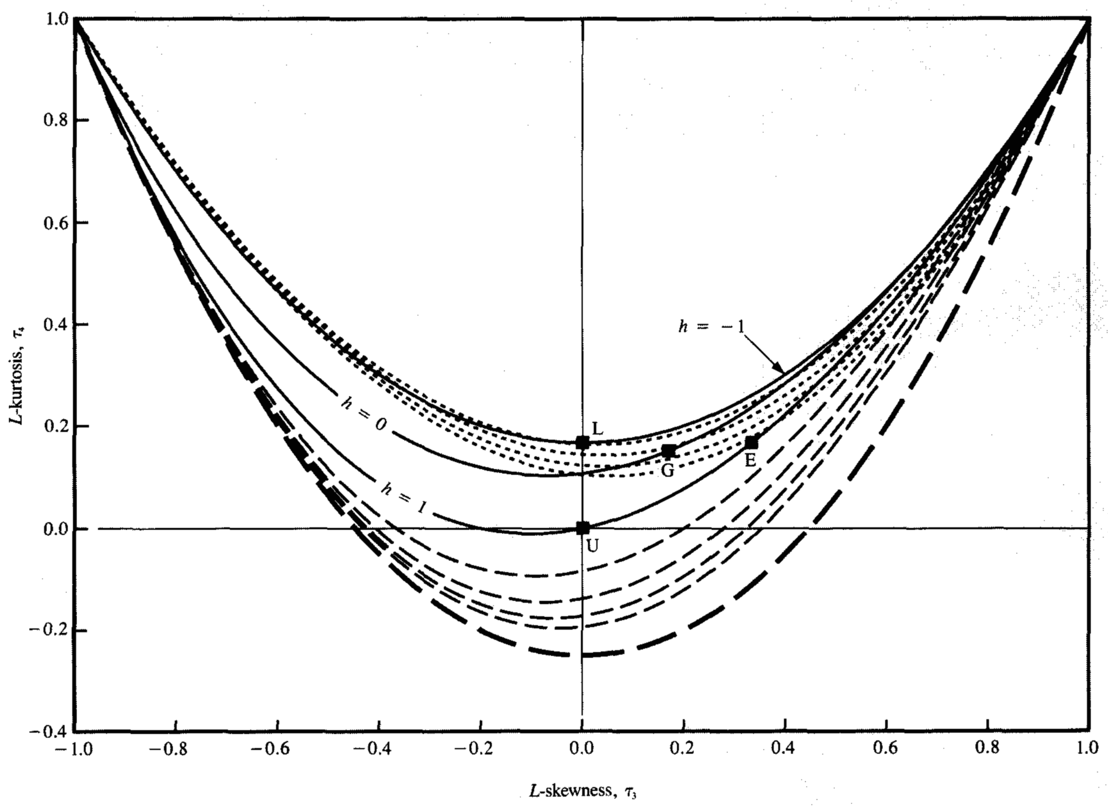

Figure 2

L-moment ratios τ_3 and τ_4 for the four-parameter kappa distribution. The graph shows τ_4 as a function of τ_3 as k varies, for fixed h. Solid lines indicate the values $h = 1$, $h = 0$, and $h = -1$. Dotted lines indicate $h = -2, -3, -4, -5$. Light dashed lines indicate $h = 2, 3, 4, 5$; the heavy dashed line is the lower bound of τ_4 for all distributions, and is attained by the four-parameter kappa distribution in the limit $h \to \infty$. The individual points marked E, G, L, and U represent the exponential, Gumbel, logistic, and uniform distributions, respectively.

to the ordinary moments: they are less biased and less affected by unusually extreme data values [21–24].

The PWMs of a distribution β_r exist if the mean of the distribution exists: i.e., for the four-parameter kappa distribution, if $h \ge 0$ and $k > -1$, or if $h < 0$ and $-1 < k < -1/h$. The β_r are most conveniently evaluated by writing (10) as

$$\beta_r = \int_0^1 x(F)F^r \, dF$$

$$= (r+1)^{-1}\left(\xi + \frac{\alpha}{k}\right) - \frac{\alpha}{k}\int_0^1 [h^{-1}(1 - F^h)]^k F^r \, dF.$$

This integral can be evaluated similarly to the way in which (9) was. We find that if $k \ne 0$, the β_r are given by

$$r\beta_{r-1} = \begin{cases} \xi + \dfrac{\alpha}{k}\left[1 - \dfrac{r\Gamma(1+k)\Gamma(r/h)}{h^{1+k}\Gamma(1+k+r/h)}\right] & \text{if } h > 0,\ k > -1, \\ \xi + \dfrac{\alpha}{k}[1 - r^{-k}\Gamma(1+k)] & \text{if } h = 0,\ k > -1, \\ \xi + \dfrac{\alpha}{k}\left[1 - \dfrac{r\Gamma(1+k)\Gamma(-k-r/h)}{(-h)^{1+k}\Gamma(1-r/h)}\right] & \text{if } h < 0,\ -1 < k < -1/h. \end{cases} \quad (12a)$$

Similarly, if $k = 0$ we have

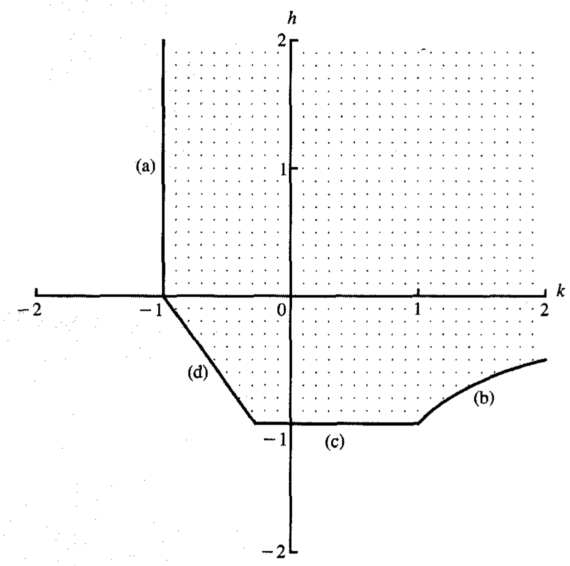

Figure 3

A parameter space (dotted area) for the h and k parameters of the four-parameter kappa distribution that ensures the existence of the first four L-moments and the uniqueness of the parameters given the first four L-moments. The letters (a), (b), (c), (d) mark lines and curves that correspond to the conditions in Equation (13).

$$r\beta_{r-1} = \begin{cases} \xi + \alpha[\gamma + \log h + \psi(1 + r/h)] & \text{if } h > 0, \\ \xi + \alpha(\gamma + \log r) & \text{if } h = 0, \\ \xi + \alpha[\gamma + \log(-h) + \psi(-r/h)] & \text{if } h < 0, \end{cases} \tag{12b}$$

where $\gamma = 0.5772 \cdots$ is Euler's constant, and ψ is the digamma function, $\psi(x) = d[\log \Gamma(x)]/dx$.

L-moments of the four-parameter kappa distribution can be obtained from (11) and (12). The L-moment ratios τ_3 and τ_4 are functions only of the shape parameters h and k. Loci of (τ_3, τ_4) as k varies are plotted for several values of h in **Figure 2**, an "L-moment-ratio diagram." It is clear that not all of the possible (τ_3, τ_4) values (i.e., points lying above the heavy dashed line in Figure 2) are generated by the four-parameter kappa distribution and that some (τ_3, τ_4) points correspond to more than one (h, k) pair.

When inference for the four-parameter kappa distribution is based on L-moments, it seems useful to restrict the parameter space so that $h > -1$. Kappa distributions so restricted can assume all (τ_3, τ_4) values between the "generalized-logistic line" ($h = -1$ in Figure 2) and the lower bound for all distributions. This should be sufficient for most practical purposes. On the L-moment-ratio diagram, the "generalized-logistic line" is the line $\tau_4 = (5\tau_3^2 + 1)/6$, and the lower bound on τ_4 is the line $\tau_4 = (5\tau_3^2 - 1)/4$ [7, 21].

Table 1 Maximum attainable value of τ_4 for specified values of τ_3 for the four-parameter kappa distribution, and the values of h and k for which this maximum is attained.

τ_3	τ_4	h	k
0.0	0.1696	−1.35	−0.041
0.1	0.1762	−1.22	−0.120
0.2	0.2002	−1.09	−0.206
0.3	0.2417	−0.96	−0.298
0.4	0.3006	−0.83	−0.394
0.5	0.3766	−0.69	−0.494
0.6	0.4695	−0.56	−0.596
0.7	0.5788	−0.41	−0.699
0.8	0.7041	−0.28	−0.802
0.9	0.8447	−0.14	−0.902

To enable the four parameters of the distribution to be estimated from the first four L-moments, the parameter space must be further restricted, so that only one set of parameters corresponds to a given set of the first four L-moments. **Table 1** gives the maximum attainable values of τ_4 for several values of τ_3, and the corresponding values of h and k. When $h > -1$, these h and k values are approximately related by $k + 0.725h = -1$. Thus, we are led to impose the following conditions on the parameters:

$$\begin{aligned} &\text{(a)}\quad k > -1; \\ &\text{(b)}\quad \text{if } h < 0, \text{ then } hk > -1; \\ &\text{(c)}\quad h > -1; \\ &\text{(d)}\quad k + 0.725h > -1. \end{aligned} \tag{13}$$

Conditions (a) and (b) ensure the existence of the L-moments; (c) and (d) ensure the uniqueness of the parameters, given the first four L-moments. The parameter space is illustrated in **Figure 3**.

Estimation of the parameters of the four-parameter kappa distribution using L-moments requires the solution of Equations (12) for the parameters, given β_0, β_1, β_2, and β_3. No explicit solution is possible, but the equations can be solved by Newton–Raphson iteration. An algorithm for this purpose has been programmed in FORTRAN and is included in [25].

Example

Jointly with M. G. Schaefer (Washington State Department of Ecology, Dam Safety Section), we analyzed data on annual maximum precipitation for sites in Washington State. Schaefer had previously analyzed [26] an earlier version of the data set. Data are available for 326 gauging sites for several durations. For example, annual maximum rainfall for a duration of two hours consists of yearly

Table 2 Description of regions used in rainfall analysis. (There were more gauging sites for the 24-hour data than for the two-hour and six-hour data.)

Duration of rainfall measurements (hr)	*Smallest region*		*Largest region*	
	Number of sites	*Number of data values*	*Number of sites*	*Number of data values*
2	13	392	5	176
6	13	392	5	176
24	41	1698	8	297

values of the greatest rainfall occurring in any two-hour interval in the year. (For example, in Spokane, the maximum rainfall for any two-hour period in 1985 was 0.31 inches.) Several distributions were fitted to annual maximum precipitations for durations of two, six, and twenty-four hours, and estimates of the quantiles of these distributions were calculated. "Regional analysis" [27, 28] was used; that is, fitting a distribution of annual maximum precipitation at one gauging site uses data not only from that site but from a "region" comprising several sites. This provides more accurate quantile estimates than would be achieved by fitting distributions to the data for each site separately. There were 13 regions, each region representing a different range of mean annual precipitation, and the sites were assigned to the appropriate regions. **Table 2** gives some information about the regions. The regional data analysis used an index-flood method based on L-moments, similar to those described in [27] and [28]. The method involves rescaling the data so that the data for each site have mean 1, using the rescaled data to calculate L-moment ratios averaged over all of the sites in a region, and fitting distributions based on the at-site mean and the regional average L-moment ratios.

The regional averages of L-skewness and L-kurtosis for the 13 regions (for each of the three durations) are shown in **Figure 4**. Figure 4 also shows the L-skewness–L-kurtosis relations for the four-parameter kappa distribution with $h = -1$, 0, or +1. Many of the regional average points lie close to the line $h = 0$, suggesting that a generalized extreme-value distribution may give a good fit to the data. This distribution was used in Schaefer's original analysis. A majority of the points for all durations lie above the $h = 0$ line, however, suggesting that a distribution intermediate between generalized extreme-value and generalized logistic may be more appropriate. The four-parameter kappa distribution is such a distribution, and it was fitted to the regional data using the algorithm given in [25]. **Figure 5** gives a typical example of the results, for two-hour-duration data for the region with sites having mean annual precipitation greater than 104 inches. The regional average L-moment ratios

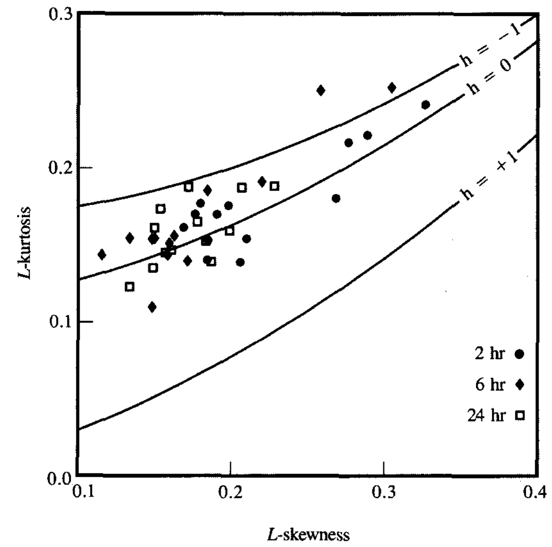

Figure 4

Regional average L-skewness and L-kurtosis of Washington State annual maximum precipitation data. Solid lines are the L-skewness–L-kurtosis relations for a four-parameter kappa distribution with $h = -1$, 0, or +1.

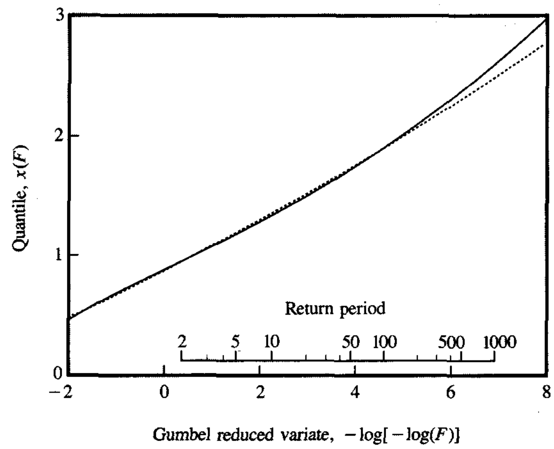

Figure 5

Distributions fitted to Washington State two-hour-duration annual maximum precipitation data for sites with mean annual precipitation greater than 104 inches: generalized extreme-value (dotted line) and four-parameter kappa distribution (solid line).

($\ell_1 = 1$, $\ell_2 = 0.1426$, $t_3 = 0.1981$, and $t_4 = 0.1758$) yield estimated four-parameter kappa distribution parameters $\xi = 0.8987$, $\alpha = 0.1764$, $k = -0.0917$, and $h = -0.2068$. The fitted GEV and four-parameter kappa distributions are shown in Figure 5. The graph is scaled according to extreme-value plotting paper, so that a Gumbel distribution would plot as a straight line. The return-period axis aids in the calibration of extreme events: an event with a return period of T years has a magnitude $x(F)$, with $F = 1 - 1/T$. The two distributions are very similar, except for high-return periods, for which the four-parameter kappa distribution with negative h has larger quantiles. Use of the four-parameter kappa distribution here, rather than the GEV, provides greater assurance that the frequency of extreme precipitations is not systematically underestimated. This is important when specifying requirements for dam safety, one of the intended applications of this study.

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Jonathan R. M. Hosking *IBM Research Division, Thomas J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598 (HOSKING at YKTVMV, hosking@watson.ibm.com).*

Dr. Hosking received an M.A. degree from Cambridge University, and a Ph.D. in statistics from the University of Southampton, England, in 1979. He has been a Research Staff Member at the Thomas J. Watson Research Center since 1986. His research interests include time-series analysis, distribution theory, and flood frequency analysis.