Discussed in this paper are asymptotic properties of the classical machine interference model, the simplest of queuing models. In systems analysis, the judicious use of such asymptotic properties can result in significant savings in time and effort.

Included in the paper is the solution of the generalized machine interference model.

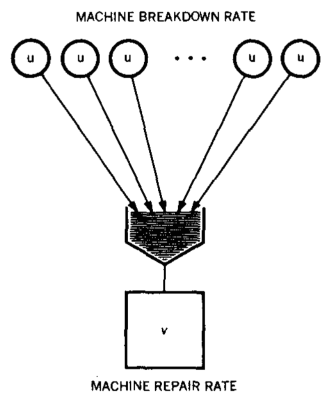
An analysis of the machine interference model

by A. E. Ferdinand

The machine interference model[1] is one of the simplest models of a queuing system, and as such, has probably become one of the most used queuing models today. In spite of this, no known significant effort has been expended in deriving the properties of this now classic model; most of the studies of the model have been mainly concerned with extensions and adaptations[2] to the model to meet certain specific needs.

Erlang employed the machine interference model in his analysis of the Swedish telephone traffic. Since that time, the model has been successfully applied in industry in general. In particular, the machine interference model is being used in the computer industry in such problems as the analysis of the flow of messages in a computer system. Because of the utility of the model, the insight into its properties that is given in this paper offers further understanding of the behavior of the model, and, consequently, use of it may be made easier.

Figure 1. The machine interference model



The machine interference model is shown in Figure 1. It consists of n identical machines and a single repairman. When a machine breaks down, it is repaired by the repairman and put back into operation. If the repairman is busy, a broken machine has to wait for service causing a queue to build up in front of the repairman. Repair and breakdown rates of the machines are assumed to be exponentially distributed. Each machine is characterized by a mean breakdown rate u and a mean repair rate v.

It would seem, at first sight, that as the number of machines, n, increases, the complexity and obscurity of the properties of the queuing system would increase tremendously. But this is not so, and in the case of a very large number of machines, new and distinct regularities appear.

asymptotic properties

In this paper, some interesting and useful asymptotic properties of the machine interference model are given. These asymptotic properties are useful in two respects. In the first instance, they give important knowledge about the actual behavior of the queuing system. With such knowledge, one can often deduce how the system is going to behave under certain specified conditions without having to perform detailed computations to come to a conclusion. At this juncture, it is well to add that knowing the exact formulas for the probability distribution, the length of queues, etc., does not necessarily mean that one understands the behavior of a model. Such an understanding can often come only through an analysis (sometimes numerical) of the exact formulas.

The asymptotic formulas are also useful in obtaining accurate approximate values of the properties of the system without significant expenditure of computation time and human effort. Invariably, as the number of machines, n, increases so does the computation time (and sometimes patience) that is necessary in the calculation of the system properties from the exact formula. However, calculations may still be performed with the asymptotic formulas with relative ease, and furthermore, with an accuracy that increases with n.

The first part of this paper will be devoted to the asymptotic properties of the machine interference model. In the second part, the solution of the generalized machine interference model will be stated but not derived. The derivation of the solution may be found in Reference 3, where the solution is only a special case of a very general solution to queuing problems.

Properties of the model

In the notation of Reference 1, the probability that the queue at the repairman is empty is given by

$$P_0 = 1/Z_n \tag{1}$$

where Z_n is the partition function given by

$$\begin{aligned} Z_n &= \sum_{k=0}^{n} \binom{n}{k} k!\, x^k \\ &= n!\, x^n \sum_{k=0}^{n} 1/(k!\, x^k) \end{aligned} \tag{2}$$

with $x = u/v$.

The utilization of the repairman or the probability that the repairman is busy is thus

$$U_n = 1 - 1/Z_n \tag{3}$$

If we now define the length of the queue at the repairman as the number of machines waiting for or receiving service at the repairman, and the length of the waiting line as the number of machines awaiting service (but not including the machine receiving service), then the mean length of the queue at the repairman is derived in the Appendix and is given by

$$\begin{aligned} L_n &= \left[\sum_{k=0}^{n} k\binom{n}{k}k!\, x^k\right]/Z_n \\ &= x\frac{d}{dx}\ln Z_n \qquad (4) \\ &= n - \frac{U_n}{x} \end{aligned}$$

and the mean length of the waiting line at the repairman is

$$\begin{aligned} W_n &= \left[\sum_{k=1}^{n} (k - 1)\binom{n}{k}k!\, x^k\right]/Z_n \\ &= L_n - U_n \qquad (5) \\ &= n - \left(1 + \frac{1}{x}\right)U_n \end{aligned}$$

The mean time required to put a broken machine back into operation, or in the terminology of computer science, the response time T_n of the system, is given by

$$\begin{aligned} T_n &= (1/v)(1 + L_n - U_n) \\ &= (1/v)\left[n + 1 - \left(1 + \frac{1}{x}\right)U_n\right] \qquad (6) \end{aligned}$$

where again v is the mean repair rate of a machine.

For large values of n, the term $k!$ in Equation 2 becomes quite large and may cause an overflow condition to be invoked when Z_n is calculated on a digital computer. On the other hand, x^k may trigger an underflow condition since x is generally positive and less than unity in realistic cases. However, if $k!$ is weighted with x^k through the definition

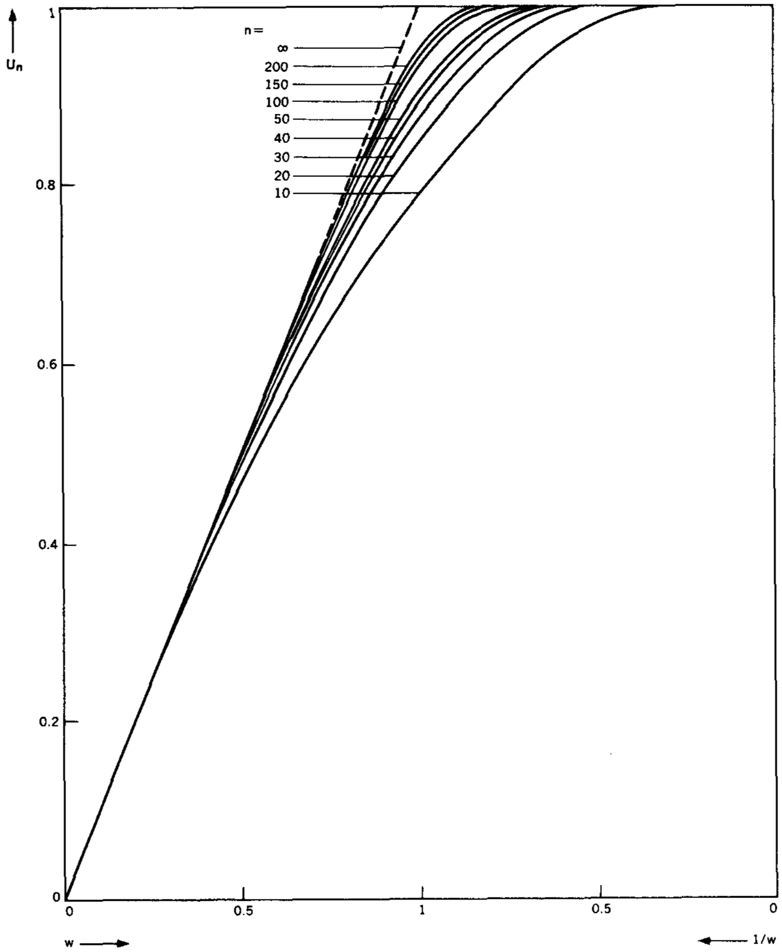
$$a(0, x) = 1 \tag{7}$$

$$a(k, x) = kxa(k - 1, x) \tag{8}$$

so that

$$Z_n = a(n, x)\sum_{k=0}^{n} 1/a(k, x) \tag{9}$$

Figure 2. The utilization U_n of the repairman



the numerical computation of Z_n by a digital computer may be carried out for considerably larger values of n without invoking either overflow or underflow conditions.

The asymptotic properties

The asymptotic formulas given in this section are all derived explicitly in the Appendix.

Figure 2 shows curves of utilization plotted against w for $w \leq 1$ and against $1/w$ for $w \geq 1$, where w is defined by

$$w = nx \tag{10}$$

The dashed line indicates the utilization of the repairman under infinite load, i.e., the utilization of the repairman when $n \to \infty$.

It is noted that as the system load n increases, the utilization converges to the utilization of the infinite system. Consequently, one can represent U_n in terms of a displacement from U_∞ which is defined by

$$\begin{aligned} U_\infty &= w \quad \text{for} \quad w \leq 1 \\ &= 1 \quad \text{for} \quad w \geq 1 \end{aligned} \tag{11}$$

One may write

$$U_n = U_\infty - f(n) \tag{12}$$

where $f(n)$, for a given w, is a positive decreasing function of n, and

$$\lim_{n\to\infty} f(n) = 0 \tag{13}$$

In realistic situations, w is usually less than 1, and the asymptotic approximation

$$U_n = w - \frac{w^2}{n(1 - w)} + \frac{2w^3}{n^2(1 - w)^3} + O\left(\frac{1}{n^3}\right) \tag{14}$$

can be used to give quick and accurate results. This formula is asymptotic in both w and n, so that for any given $w < 1$, the accuracy increases with increasing n, and for any given n, the accuracy increases as w decreases. The asymptotic formulas for U_n when $w \geq 1$ are given later in this section and also in the Appendix.

With the aid of Equations 3, 4, and 14, one can show that the mean length of the queue at the repairman is an extensive property of the total number of machines n. It is therefore prudent to study a normalized queue length, i.e., the queue length per machine. Figure 3 shows the normalized queue length L, defined by

$$L = L_n/n \tag{15}$$

plotted against w for $w \leq 1$ and against $1/w$ for $w \geq 1$. The dashed line shows the normalized length of the queue in the limit as the number of machines $n \to \infty$. It is given by

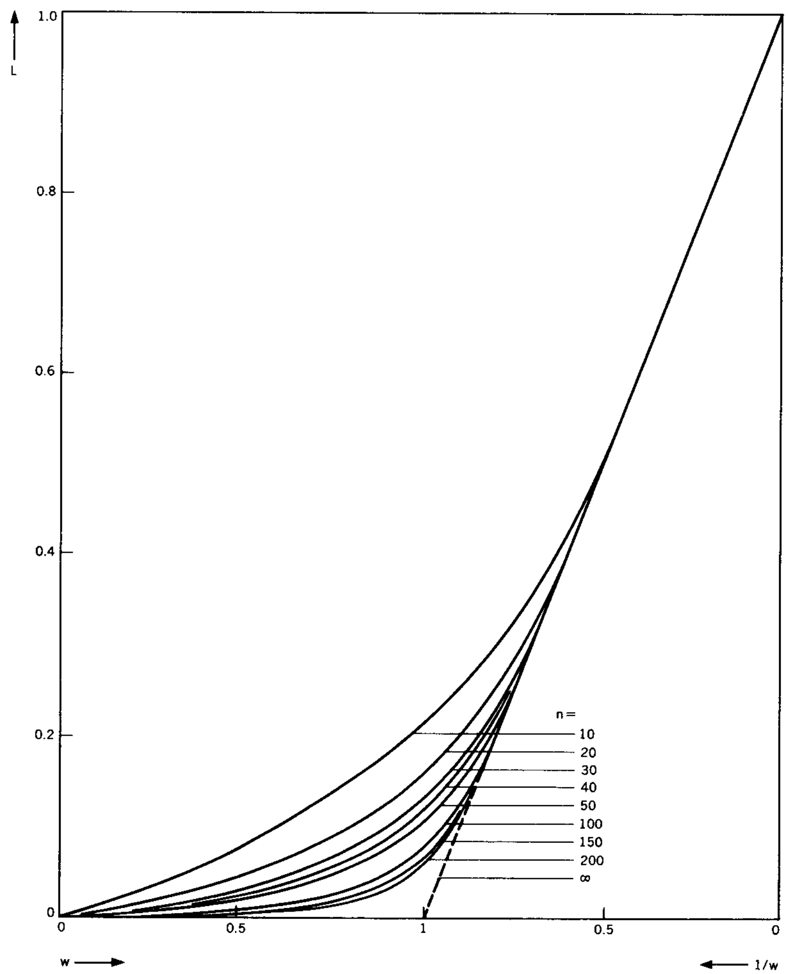
$$\begin{aligned} L(w) &= \lim_{n\to\infty} L \\ &= 0 \qquad\qquad \text{for} \quad w \leq 1 \\ &= 1 - 1/w \quad \text{for} \quad w \geq 1 \end{aligned} \tag{16}$$

Here again, we can observe the convergence to the limiting curve $L(w)$. The asymptotic formula for L_n is given by

$$L_n = \left(\frac{w}{1 - w}\right)\left[1 - \frac{2w}{n(1 - w)^2}\right] + O\left(\frac{1}{n^2}\right) \qquad \text{for} \quad w < 1 \tag{17}$$

For any given $w < 1$, the accuracy of Equation 17 increases with increasing n, and for any given n, accuracy increases with decreasing

Figure 3. The normalized length of the queue $L = L_n/n$



w. For $w > 1$, the convergence is quite rapid and for all practical purposes, we may write

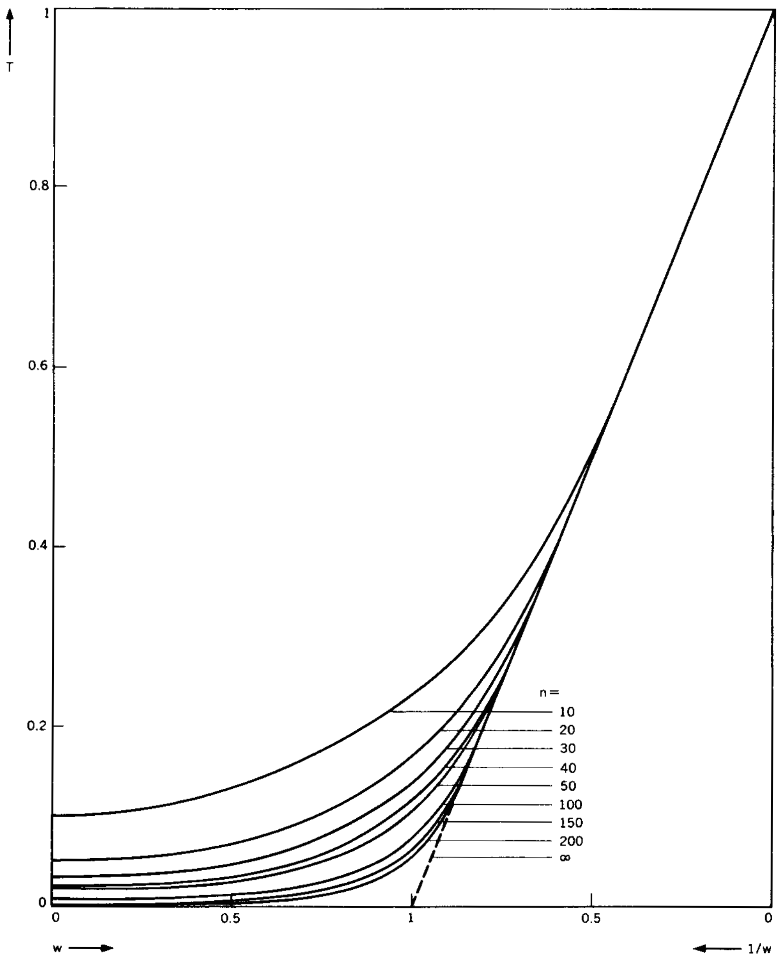
$$L_n \approx n\left\{1 - \frac{1}{w}\left[1 - \left(\frac{1}{w}e^{1-1/w}\right)^n \Big/ (2\pi n)^{1/2}\right]\right\} \quad \text{for } w > 1 \tag{18}$$

The variable w may be designated the system variable since it is the variable in which the properties of the system can be most naturally described.

The asymptotic behavior of the length of the waiting line W_n and the response time T_n of the system can be obtained by use of the asymptotic formulas given for L_n and U_n. Figure 4 shows the doubly normalized response time plotted against w for $w \leq 1$, and against $1/w$ for $w \geq 1$. This doubly normalized response time T is defined by

$$T = vT_n/n \tag{19}$$

Figure 4. The double normalized response time $T = v\,T_n/n$



In summary, the asymptotic formulas are listed here. A further display of the accuracy of the approximations is shown in Table 1 where some results obtained through the use of the asymptotic formulas are compared with the exact values. Again, these asymptotic formulas are all derived explicitly in the Appendix.

The asymptotic formulas for $w < 1$ are

$$U_n = w - \frac{w^2}{n(1-w)} + \frac{2w^3}{n^2(1-w)^3} + O\left(\frac{1}{n^3}\right) \qquad (20)$$

$$L_n = \left(\frac{w}{1-w}\right)\left[1 - \frac{2w}{n(1-w)^2}\right] + O\left(\frac{1}{n^2}\right) \qquad (21)$$

$$W_n = L_n - U_n = n\left[1 - \left(\frac{1}{n} + \frac{1}{w}\right)U_n\right] \qquad (22)$$

$$T_n = (1/v)[1 + L_n - U_n] = (n/v)\left[1 + \frac{1}{n} - \left(\frac{1}{n} + \frac{1}{w}\right)U_n\right] \qquad (23)$$

Table 1 Comparison of asymptotic values and exact values

n	w	Asymptotic values U_n	Asymptotic values L_n/n	Exact values U_n	Exact values L_n/n
10	0.05	0.04974	0.00520	0.04974	0.00521
10	0.2	0.19531	0.02344	0.19528	0.02358
50	0.05	0.04995	0.00105	0.04995	0.00105
50	0.2	0.19901	0.00494	0.19901	0.00494
50	0.5	0.49080	0.01840	0.49070	0.01861
100	0.2	0.19950	0.00248	0.19950	0.00248
100	0.5	0.49520	0.00960	0.49519	0.00963
150	0.5	0.49676	0.00649	0.49675	0.00650

where L_n and U_n are given in Equations 21 and 20 respectively. For $w > 1$

$$U_n \approx 1 - \left(\frac{1}{w} e^{1-1/w}\right)^n \Big/ (2\pi n)^{1/2} \tag{24}$$

$$L_n \approx n\left\{1 - \frac{1}{w}\left[1 - \left(\frac{1}{w} e^{1-1/w}\right)^n \Big/ (2\pi n)^{1/2}\right]\right\} \tag{25}$$

The mean length of the waiting line and response time for $w > 1$ can be obtained from Equations 22 and 23 respectively with L_n and U_n given in Equations 25 and 24 respectively.

At the point $w = 1$, the asymptotic formulas are

$$U_n = 1 - \left(\frac{2}{\pi n}\right)^{1/2} + \frac{4/3}{\pi n} + O\left(\frac{1}{n^{3/2}}\right) \tag{26}$$

$$L_n = \left(\frac{2n}{\pi}\right)^{1/2} - \frac{4}{3\pi} + O\left(\frac{1}{n^{1/2}}\right) \tag{27}$$

the unstable point

The partition function of the machine interference model with a finite number of machines is a finite series. As such, it is an analytic function of x and contains no singular points. But, in the limit as the number of machines $n \rightarrow \infty$, the partition function develops a mathematical singularity at the point $w = 1$. This is amply demonstrated in Figures 2, 3, and 4 by the kinks in the utilization, queue length, and response time curves at the point $w = 1$. Convergence to the limiting curve is also very slow around the point $w = 1$. For finite n, dL/dx goes through a maximum at a point $w_{\max}$ where

$$w_{\max} = 1 + g(n) \tag{28}$$

$$\frac{dL}{dx} \sim Cn \quad \text{at} \quad w_{\max} \tag{29}$$

$g(n)$ being a positive decreasing function of n and C a constant. The point $w_{\max}$ may be called the unstable point of the system, and it tends to the critical point as $n \rightarrow \infty$. The function dL/dx diverges to infinity with n at the critical point as $n \to \infty$. One will find in performing a simulation around the point w_{max} that because of the large fluctuations in that vicinity, the time taken to attain equilibrium will be longer than it will be in regions away from w_{max}. Furthermore, this difficulty to attain equilibrium around w_{max} will also increase as n is increased. The simulation model and the mathematical queuing model may sometimes be found to give different answers in the vicinity of w_{max}. This is because the mathematical model always represents the system under equilibrium conditions, and the simulation model may not have attained equilibrium even after a presumably long settling time. Contrary to commonly held belief, the utilization has nothing to do with this disagreement between the simulation and mathematical models, and it is only incidental that the utilization is high in the region around w_{max}.

The generalized machine interference model

The generalized system consists of n machines and a single repairman, and here again, the breakdown and repair rates of the machines are assumed to be exponentially distributed. The kth machine is characterized by a mean breakdown rate u_k, a mean repair rate v_k, and by an occupational variable d_k defined by

$d_k = 0$ if the kth machine is broken and is awaiting repair or being repaired

$d_k = 1$ if the kth machine is operational (30)

The derivation of the solution to this problem will not be repeated in this paper, but appears as a simple special case of a general solution to queuing problems in Reference 3.

The partition function for this generalized machine interference model may be written as

$$Z_n = \sum_{d_1} \sum_{d_2} \cdots \sum_{d_n} \left[\sum_{k=1}^{n} (1 - d_k) \right]! \prod_{k=1}^{n} x_k^{1-d_k} \tag{31}$$

where

$$x_k = u_k/v_k \tag{32}$$

and the probability of being in a state characterized by a vector $(d_1, d_2, \cdots, d_n)$ is

$$p(d_1, d_2, \cdots, d_n) = \left\{ \left[\sum_{k=1}^{n} (1 - d_k) \right]! \prod_{k=1}^{n} x_k^{1-d_k} \right\} / Z_n \tag{33}$$

with Z_n given in Equation 31. The classical queuing equation that this probability distribution function satisfies is given (but not derived) in the Appendix.

When the queuing system is in a state characterized by a vector $(d_1, d_2, \cdots, d_n)$, the queue at the repairman has a length $\sum_{k=1}^{n} (1 - d_k)$, so that the mean length of the queue L_n is given by

$$\begin{aligned} L_n &= \sum_{d_1} \sum_{d_2} \cdots \sum_{d_n} p(d_1, d_2, \cdots, d_n) \sum_{k=1}^{n} (1 - d_k) \\ &= \sum_{k=1}^{n} x_k \frac{d}{dx_k} \ln Z_n \end{aligned} \tag{34}$$

which is the sum of the mean values of $(1 - d_k)$ for $k = 1, 2, \cdots, n$. From Equation 33, the probability that the repairman is idle is

$$p(1, 1, \cdots, 1) = 1/Z_n \tag{35}$$

so that the utilization of the repairman is simply

$$U_n = 1 - 1/Z_n \tag{36}$$

When the kth machine is broken, the mean length of time T_k required to put it back into operation is given by

$$T_k = 1/v_k + (1 - U_n/L_n) \sum_{l=0}^{n} (1/v_l) x_l \frac{d}{dx_l} \ln Z_n \tag{37}$$

One may check that in the special case when all machines possess the same characteristics, each formula given for this generalized model reduces to the equivalent formula for the ordinary machine interference model.

The occupational variable d_k makes the task of evaluating the utilization of any given machine or combination of machines a simple one indeed. The utilization $U_n^{(r)}$ of the rth machine in the generalized model is the probability that the occupational variable $d_r = 1$. That is,

$$\begin{aligned} U_n^{(r)} &= \Bigg\{ \sum_{d_1} \sum_{d_2} \cdots \sum_{d_{r-1}} \sum_{d_r = 1} \sum_{d_{r+1}} \cdots \\ &\qquad \times \sum_{d_n} \left[\sum_{k=1}^{n} (1 - d_k) \right]! \prod_{k=1}^{n} x_k^{1-d_k} \Bigg\} \Big/ Z_n \\ &= Z_{n-1}^{(r)}/Z_n \end{aligned} \tag{38}$$

where

$$Z_{n-1}^{(r)} = \sum_{d_1} \sum_{d_2} \cdots \sum_{d_{r-1}} \sum_{d_{r+1}} \cdots \sum_{d_n} \left[{\sum_{k=1}^{n}}' (1 - d_k) \right]! {\prod_{k=1}^{n}}' x_k^{1-d_k} \tag{39}$$

and where $\sum'$ and $\prod'$ indicate that the rth term is omitted in the sum and product over k. $Z_{n-1}^{(r)}$ represents the partition function of a generalized model in which the rth machine is omitted. In general, the utilization of a specific set $[s]$ of s machines is given by

$$U_{n-s}^{[s]} = Z_{n-s}^{[s]}/Z_n \tag{40}$$

where $Z_{n-s}^{[s]}$ is the partition function for the generalized model consisting of the complementary set $[n - [s]]$ of machines which does not include the set $[s]$.

Summary

The asymptotic properties of the machine interference model were described in this paper. When the properties of the system were appropriately normalized and represented in terms of the variable $w\ (=nu/v)$, the size dependencies of these properties were of order $1/n$ in regions of the most interest, that is, $w < 1$. Furthermore, the model displayed a functionally different behavior in the region $w \geq 1$. The region around $w = 1$ was described as an unstable region in which an equivalent simulation model could be expected to stabilize slowly and probably give results different from the mathematical model. The asymptotic formulas are simple in form and quite accurate, and they allow the performance of queuing analysis with very little effort.

In the last section of the paper, the solution and the properties of the generalized machine interference model were given.

ACKNOWLEDGMENT

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Appendix

steady-state queuing equation

The steady-state queuing equation for the generalized machine interference model is given by

$$p(d_1, d_2, \cdots, d_n)\left\{\sum_{k=1}^{n} u_k d_k + \left[1\Big/\sum_{k=1}^{n}(1-d_k)\right]\sum_{k=1}^{n} v_k(1-d_k)\right\}$$
$$= \sum_{k=1}^{n} u_k p(d_1, d_2, \cdots, d_{k-1}, d_k + 1, d_{k+1}, \cdots, d_n)$$
$$+ \left\{1\Big/\left[1 + \sum_{k=1}^{n}(1-d_k)\right]\right\}$$
$$\times \sum_{k=1}^{n} v_k p(d_1, d_2, \cdots, d_{k-1}, d_k - 1, d_{k+1}, \cdots, d_n)$$

where

$$\sum_{k=1}^{n} d_k \leq n$$

$$p(d_1, d_2, \cdots, d_{k-1}, d_k + 1, d_{k+1}, \cdots, d_n) = 0 \quad \text{if} \quad d_k = 1$$

$$p(d_1, d_2, \cdots, d_{k-1}, d_k - 1, d_{k+1}, \cdots, d_n) = 0 \quad \text{if} \quad d_k = 0$$

The probability distribution function that satisfies the steady-state equation shown above is given in Equation 33.

The partition function Z_n of the machine interference model can be written in the form

$$Z_n = e^{1/x}x^n\Gamma(n+1)[1-\gamma(n+1, 1/x)/\Gamma(n+1)] \tag{A1}$$

where $\Gamma(n+1)$ is the Gamma Function, and $\gamma(n+1, 1/x)$ is the Incomplete Gamma Function.[4] If we now write

$$\frac{1}{x} = ns \tag{A2}$$

then Equation A1 can be put in the form

$$Z_n = (se^{1-s})^{-n}[1/\beta(n)][1-\gamma(n+1, ns)/\Gamma(n+1)] \tag{A3}$$

where

$$\beta(n) = e^{-n}n^n/\Gamma(n+1) \tag{A4}$$

The ratio of the Gamma Functions can be written as[5]

$$\begin{aligned}
&\gamma(n+1, ns)/\Gamma(n+1)\\
&= \left(\frac{s}{1-s}\right)(se^{1-s})^n\beta(n)\left[1 - \frac{1}{n(1-s)^2} + \frac{1+2s}{n^2(1-s)^4} + O\left(\frac{1}{n^3}\right)\right] \quad \text{for} \quad s < 1\\
&= \frac{1}{2} - \frac{2/3}{(2\pi n)^{1/2}} + O\left(\frac{1}{n^{3/2}}\right) \quad \text{for} \quad s = 1\\
&= 1 + \left(\frac{s}{1-s}\right)(se^{1-s})^n\beta(n)\\
&\times\left[1 - \frac{1}{n(1-s)^2} + \frac{1+2s}{n^2(1-s)^4} + O\left(\frac{1}{n^3}\right)\right] \quad \text{for} \quad s > 1
\end{aligned} \tag{A5}$$

By now making the substitution

$$s = 1/w \tag{A6}$$

where we recall that $w = nu/v$, the partition function can be put in its final form

$$\begin{aligned}
Z_n &= \left(\frac{1}{1-w}\right)\left[1 - \frac{w^2}{n(1-w)^2} + \frac{w^4+2w^3}{n^2(1-w)^4} + O\left(\frac{1}{n^3}\right)\right] \quad \text{for} \quad w < 1\\
&= [1/\beta(n)]\left[\frac{1}{2} + \frac{2/3}{(2\pi n)^{1/2}} + O\left(\frac{1}{n^{3/2}}\right)\right] \quad \text{for} \quad w = 1\\
&= \left(\frac{1}{w}e^{1-1/w}\right)^{-n}(1/\beta(n)) - \frac{1/w}{(1-1/w)} + O\left(\frac{1}{n}\right) \quad \text{for} \quad w > 1
\end{aligned} \tag{A7}$$

By using the relation

$$1/\beta(n) = (2\pi n)^{1/2}\left(1 + \frac{1}{12n}\right) + O(n^{-3/2}) \tag{A8}$$

the utilization U_n then becomes

$$\begin{aligned} U_n &= 1 - 1/Z_n \\ &= w - \frac{w^2}{n(1-w)} + \frac{2w^3}{n^2(1-w)^3} + O\left(\frac{1}{n^3}\right) \quad \text{for} \quad w < 1 \\ &= 1 - \left(\frac{2}{\pi n}\right)^{1/2} + \frac{4/3}{\pi n} + O\left(\frac{1}{n^{3/2}}\right) \quad \text{for} \quad w = 1 \\ &= 1 - \left(\frac{1}{w}e^{1-1/w}\right)^n \Big/ (2\pi n)^{1/2} \quad \text{for} \quad w > 1 \end{aligned} \tag{A9}$$

The mean length of the queue L_n is given by

$$L_n = x\frac{d}{dx}\ln Z_n \tag{A10}$$

By using the relation[4]

$$x\frac{d}{dx}\gamma\left(n+1, \frac{1}{x}\right) = -\frac{1}{x}\left(\frac{1}{x}\right)^n e^{-1/x} \tag{A11}$$

we then obtain

$$\begin{aligned} L_n &= n - \frac{1}{x} + (1/x) \Big/ \left\{x^n e^{1/x}\Gamma(n+1) \right. \\ &\quad \left. \times \left[1 - \gamma\left(n+1, \frac{1}{x}\right) \Big/ \Gamma(n+1)\right]\right\} \\ &= n - \frac{1}{x}(1 - 1/Z_n) \\ &= n - U_n/x \\ &= n(1 - U_n/w) \end{aligned} \tag{A12}$$

By substituting the asymptotic formula for U_n into Equation A12, we obtain the asymptotic formula for the mean length of the queue as

$$\begin{aligned} L_n &= \left(\frac{w}{1-w}\right)\left[1 - \frac{2w}{n(1-w)^2}\right] + O\left(\frac{1}{n^2}\right) \quad \text{for} \quad w < 1 \\ &= \left(\frac{2n}{\pi}\right)^{1/2} - \frac{4}{3\pi} + O\left(\frac{1}{n^{1/2}}\right) \quad \text{for} \quad w = 1 \\ &\approx n\left\{1 - \frac{1}{w}\left[1 - \left(\frac{1}{w}e^{1-1/w}\right)^n \Big/ (2\pi n)^{1/2}\right]\right\} \quad \text{for} \quad w > 1 \end{aligned} \tag{A13}$$

The mean length of the waiting line W_n is given by

$$W_n = L_n - U_n = n\left[1 - \left(\frac{1}{n} + \frac{1}{w}\right)U_n\right] \tag{A14}$$

by Equation A12. The asymptotic formula for W_n can be obtained by substituting the value of U_n given in Equation A9.

Finally we obtain the derivative of L_n with respect to x. It is given by

$$\begin{aligned}\frac{d}{dx} L_n &= \frac{d}{dx}\left[n - \frac{1}{x}\left(1 - \frac{1}{Z_n}\right)\right] \\ &= \frac{1}{x^2}\left(1 - \frac{1}{Z_n}\right) - \frac{1}{xZ_n}\frac{d}{dx}\ln Z_n \\ &= \frac{1}{x^2}\left(U_n - L_n/Z_n\right) \\ &= \left(\frac{n}{w}\right)^2\left[U_n - \frac{n}{Z_n}\left(1 - \frac{U_n}{w}\right)\right]\end{aligned} \qquad \text{(A15)}$$

At the point $w = 1$, $(d/dx)L_n$ has the asymptotic form

$$\frac{d}{dx} L_n = n^2\left[\left(1 - \frac{2}{\pi}\right) - \left(\frac{2}{\pi n}\right)^{1/2}\left(1 - \frac{8}{3\pi}\right) + O\left(\frac{1}{n}\right)\right] \qquad \text{(A16)}$$

CITED REFERENCES

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