NUMERICAL METHODS FOR THE FIRST BIHARMONIC EQUATION AND FOR THE TWO-DIMENSIONAL STOKES PROBLEM

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STAN-CS-77-615
MAY 1977

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

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We describe in this report various methods, iterative and "almost direct," for solving the first biharmonic problem on general two-dimensional domains once the continuous problem has been approximated by an appropriate mixed finite element method. Using the approach described in this report we recover some well known methods for solving the first biharmonic equation as a system of coupled harmonic equations, but some of the methods discussed here are completely new, including a conjugate gradient type algorithm. In the last part of this report we discuss the extension of the above methods to the numerical solution of the two dimensional Stokes problem in p- connected domains $(p \geq 1)$ through the stream function-vorticity formulation.

CONTENTS

ACKNOWLEDGEMENTS

A part of the results in this Report have been obtained while the first author was wisiting the Computer Sciences Department of Stanford University with the support of ERDA grant E(04-3) 326 PA #30.

We would like to thank Pr. P.G. CIARLET and G.H. GOLUB for their interest in this work.

1. INTRODUCTION.

Throughout this paper Ω denotes an open set of $\mathbb{R}^2$ of boundary Γ. Given three functions f, g_1, g_2, we shall consider the Dirichlet problem for the biharmonic operator :

$$(P_o) \quad \begin{cases} \Delta^2\psi = f \\ \psi|_\Gamma = g_1 \\ \dfrac{\partial\psi}{\partial n} = g_2 \end{cases}$$

This problem arises in fluid mechanics and in solid mechanics (bending of elastic plates).

In fluid mechanics the stream functions ψ of incompressible flows in $\mathbb{R}^2$, at low Reynolds number, is the solution of a problem (P_o), provided that Ω is simply connected. If Ω is multi-connected, ψ satisfies also a biharmonic equation but the boundary conditions are more complicated (see Sections 6 and 7). In $\mathbb{R}^3$, for axisymmetric flows, ψ is the solution of a Dirichlet problem for an operator E^2 where E is an elliptic operator of order 2, (see HAPPEL-BRENNER [29]) ; however the method to be described can be easily adapted to this situation.

For incompressible flows at large Reynolds numbers, described by the Navier-Stokes equations, a good code for the numerical solution of (P_o) is of great practical interest because many iterative techniques for the resolution of the Navier Stokes equation requires the numerical solution of a cascade of biharmonic problems like (P_o). This is clearly shown in FIX [18], [19], ROACHE [41], ROACHE-ELLIS [42] for the 2 dimensional case. Generalization of the following ideas can also lead to codes for solving the 3 dimensional Navier-Stokes equation (GLOWINSKI-PIRONNEAU [24]) and for multiconnected bidimensional domains (see Sections 6 and 7).

Finite difference discretization of (P_o) are not feasible in many cases, namely when the geometry of Ω is complicated. Standard finite element methods for solving (P_o) require rather sophisticated finite elements such as the 21-degree-of-freedoms of ARGYRIS (see ARGYRIS-DUNNE [11]) or non conforming elements of Hermite type.

Recently a new class of methods, called mixed methods has been proved to be quite appropriate to the biharmonic operator (CIARLET-RAVIART [10], BREZZI-RAVIART [6], ODEN [37]). Their drawbacks lie in the fact that they require the solution of rather large non-symmetrical linear systems. Our method is closely related to the mixed methods but its implementation is quite different and much easier. In the continuous case the underlying idea of the method can be outlined as follow :

If ψ_o denotes the solution of

$$\Delta^2\psi_0 = f \text{ in } \Omega,\ \Delta\psi_o|_\Gamma = 0\ ,\ \psi_o|_\Gamma = g_1,$$

then $\psi-\psi_o$ is the solution of (P_o) with f=0, g_1=0, and g_2 replaced by $g_2 - \frac{\partial\psi_o}{\partial n}$. Therefore from now on we assume that f = 0, $g_1 = 0$.

Let $\omega = -\Delta\psi$ and suppose that $\lambda = \omega|_\Gamma$ is known. Then (P_o) splits into two-Dirichlet problems for $-\Delta$:

$$(1.1)\quad \begin{cases} -\Delta\omega = 0 \text{ in } \Omega, \\ \omega|_\Gamma = \lambda\ , \end{cases} \qquad (1.2)\quad \begin{cases} -\Delta\psi = \omega \text{ in } \Omega, \\ \psi|_\Gamma = 0. \end{cases}$$

Let A denote the linear operator $\lambda \to -\frac{\partial\psi}{\partial n}|_\Gamma$, where ψ is computed by (1.1), (1.2). Then we shall show that the solution of (P_o) is the solution of (1.1), (1.2) with A solution of the linear problem

$$(E_o)\qquad AX = -g_2\ ,$$

More precisely it can be shown that the solution of (1.1), (1.2), (E_o) is the solution of a mixed variational formulation of (P_o). Furthermore A is symmetric positive definite, strongly elliptic from the boundary Sobolev space $H^{-1/2}(\Gamma)$ to the Sobolev space $H^{1/2}(\Gamma)$. This last property is numerically very important, provided it is preserved by the discretization, because it insures that (E_o) is a well behaved linear system. From the theoretical point of view it means also that (E_o) is an integral formulation on Γ equivalent to (P_o).

The feasibility of the method relies entirely on the ellipticity of A. Thus beside the statement of the method, the main purpose of the paper is to show that A is a symmetric positive definite operator on the Sobolev space $H^{-1/2}(\Gamma)$ and that the nice properties of A are preserved by the finite element discretisation. The proofs use a mixed formulation of (P_o) equivalent to (1.1), (1.2), (E_o). Therefore (1.1), (1.2), (E_o) is also a nice way of solving the mixed formulation of the biharmonic problem. This remark provides us with an error estimate for the method (Section 3.3).

Unless (P_o) is to be solved many times for different f and g's it is much faster to use a conjugate gradient method for the resolution of the discrete analogue of (E_o).

Historically, the decomposition of (P_o) into (1.1) and (1.2) is known in fluid mechanics. Quite a few paper have made use of it ; among others let us mention SMITH [44],[45],[46], BOSSAVIT [4], EHRLICH [14],[15],[16], Mc LAURIN [34], EHRLICH-GUPTA [17], GREENSPAN-SCHULTZ [28]. However these works are related to finite differences approximations on rectangles and are not using the fact that the discretized problem is equivalent to a linear system, related to the discrete trace of $-\Delta\psi$, whose matrix is positive definite. We have also the feeling that our approach answers some of the questions arising in FIX [18], [19]. Thus to our knowledge, most of the methods to be described are new.

Numerical experimentations have been done to test the methods described later ; the corresponding results will be published elsewhere. However some indications will be given in Sec. 8.

2. THE CONTINUOUS PROBLEM

2.1. Functional background and notations.

The following linear spaces play a fundamental part in the study of the continuous problem :

$$H^2(\Omega) = \{v \mid v \in L^2(\Omega), \frac{\partial v}{\partial x_i} \in L^2(\Omega), \frac{\partial^2 v}{\partial x_i \partial x_j} \in L^2(\Omega), 1 \leq i,j \leq 2\},$$

$$V = H^2(\Omega) \cap H^1_o(\Omega) = \{v \in H^2(\Omega) \mid v|_\Gamma = 0\},$$

$$H^2_o(\Omega) = \{v \in H^2(\Omega) \mid v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma\}.$$

The space $H^2(\Omega)$ is a Hilbert space for the scalar product :

$$(u,v)_{H^2(\Omega)} = (u,v)_{L^2(\Omega)} + \sum_{i=1,2} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_{L^2(\Omega)} + \sum_{i,j=1,2} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right)_{L^2(\Omega)}.$$

If Ω is bounded and its boundary Γ is smooth one can show the following :

Proposition 2.1. : The mapping $v \to \|\Delta v\|_{L^2(\Omega)}$ defines a norm on V equivalent to the norm induced by $H^2(\Omega)$. ∎

We shall also use the following spaces :

$$H(\Omega;\Delta) = \{v \in L^2(\Omega) \mid \Delta v \in L^2(\Omega)\},$$

$$\mathcal{H} = \{v \in H(\Omega;\Delta) \mid \Delta v = 0\}.$$

The space $H(\Omega;\Delta)$ is a Hilbert space with the scalar product

$$(u,v)_{H(\Omega;\Delta)} = (u,v)_{L^2(\Omega)} + (\Delta u, \Delta v)_{L^2(\Omega)}.$$

The norm associated with it is

$$(2.1) \qquad \|v\|_{H(\Omega;\Delta)} = (\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)})^{1/2}.$$

From (2.1) it is easy to show the

Proposition 2.2. : On $\mathcal{H}$ the topologies induced by $H(\Omega;\Delta)$ and $L^2(\Omega)$ are identical.

2.2. Traces properties

Let γ_o, γ_1 be the following trace mappings :

$$\gamma_o v = v|_\Gamma \ , \ \gamma_1 v = \frac{\partial v}{\partial n}|_\Gamma \ .$$

The following results are shown in LIONS-MAGENES [32] and the references therein :

Proposition 2.3.: The mapping $\{\gamma_o,\gamma_1\}$ is linear continuous and onto $H^{-1/2}(\Gamma)\times H^{-3/2}(\Gamma)$ from $H(\Omega;\Delta)$.

Proposition 2.4. : The mapping $\{\gamma_o,\gamma_1\}$ is linear continuous and onto $H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ from $H^2(\Omega)$.

Proposition 2.5. : The mapping γ_1 is onto $H^{1/2}(\Gamma)$ from V.

Proposition 2.6. : Restricted to $\mathcal{H}$ the mapping γ_o is an isomorphism (topological and algebraical) from $\mathcal{H}$ onto $H^{-1/2}(\Gamma)$.

2.3. Green's formula

We shall denote by $\langle\cdot,\cdot\rangle$ (resp. $\langle\langle\cdot,\cdot\rangle\rangle$) the bilinear form of the duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ (resp. $H^{3/2}(\Gamma)$ and $H^{-3/2}(\Gamma)$) which extends $(\cdot,\cdot)_{L^2(\Gamma)}$, i.e. $\langle v,w\rangle = \int_\Gamma vwd\Gamma \ \forall v \in H^{1/2}(\Gamma)$, $w \in L^2(\Gamma)$ (resp. $\langle\langle v, w\rangle\rangle = \int_\Gamma vwd\Gamma \ \forall v \in H^{3/2}(\Gamma)$, $w \in L^2(\Gamma)$). Then Green's formula (see [32]) is written

$$(2.2)\quad \left| \begin{array}{l} \displaystyle\int_\Omega v\Delta udx - \int_\Omega u\Delta vdx = \langle\gamma_1 u,\gamma_o v\rangle - \langle\langle\gamma_0 u,\gamma_1 v\rangle\rangle \\ \forall u \in H^2(\Omega) \ , \forall v \in H(\Omega;\Delta) \ . \end{array} \right.$$

2.4. Existence, unicity and decomposition results for (P_o).

Let us assume that in (P_o) one has

$$(2.3) \qquad f \in L^2(\Omega)\ ,\ g_1 \in H^{3/2}(\Gamma),\ g_2 \in H^{1/2}(\Gamma).$$

From [32] we have the

Theorem 2.1. : Problem (P_o) has one and only one solution in $H^2(\Omega)$. ■

Then it is easy to show the

Proposition 2.7. : Problem (P_o) is equivalent to

$$(2.4) \qquad \left| \begin{array}{l} -\Delta\omega = f, \\ -\Delta\psi = \omega, \\ \gamma_o\psi = g_1,\ \gamma_1\psi = g_2. \end{array} \right.$$

Remark 2.1. : The decomposition (2.4) is well known in fluid mechanics: ω is the vorticity and ψ the stream function. ■

In the following the trace of ω on Γ will play a key role, both theoretically and numerically.

Proposition 2.8 : If conditions (2.3) on f, g_1, g_2 hold, then ω admits a trace $\gamma_o\omega \in H^{-1/2}(\Gamma)$.

Proof : Since $\psi \in H^2(\Omega)$, $\omega = -\Delta\psi \in L^2(\Omega)$ and from (2.4), $\Delta\omega = -f \in L^2(\Omega)$. Therefore $\omega \in H(\Omega;\Delta)$ and from Proposition 2.3 , $\gamma_o\omega \in H^{-1/2}(\Gamma)$.

2.5. Study of the relation between $\gamma_o\omega$ and $\gamma_1\psi$

A few iterative schemes for the numerical solution of (P_o) (see [9], BOURGAT [5], GLOWINSKI [21], Sec. 5 below...) as well as the quasi-direct method below are in fact based upon the results of this section. In this direction Lemma 2.1. below is essential.

Lemma 2.1 : Let $\lambda \in H^{-1/2}(\Gamma)$ then the following holds ;

(i) The problem

$$(2.5)\quad \begin{cases} \Delta^2\psi = 0 \\ \psi|_\Gamma = 0 \\ -\Delta\psi|_\Gamma = \lambda \end{cases}$$

has a unique solution in $V = H^2(\Omega) \cap H_o^1(\Omega)$.

(ii) If ψ is the solution of (2.5) in V, the (unbounded) operator A defined by

$$(2.6)\qquad AX = -\gamma_1\psi$$

is an isomorphism from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$.

(iii) The bilinear form $a : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R}$ defined by

$$(2.7)\qquad a(\lambda,\mu) = \langle A\lambda,\mu\rangle$$

is continuous, symmetric and $H^{-1/2}(\Gamma)$-elliptic.

Proof of (i) : The variational problem

$$(2.8)\quad \begin{cases} \int_\Omega \Delta\psi\Delta v dx = -\langle \frac{\partial v}{\partial n}, \lambda\rangle \ \forall v \in V, \\ \psi \in V \quad , \end{cases}$$

has one and only one solution. This result is classical. Nevertheless let us prove it : the domain Ω being bounded and the boundary Γ of Ω being regular, $||\Delta v||_{L^2(\Omega)}$ defines on V a norm equivalent to the norm induced by $H^2(\Omega)$ (see Proposition 2.1.). Therefore the bilinear form

$$(u,v) \to \int_\Omega \Delta u\Delta v dx$$

is continuous on $V \times V$ and V-elliptic. The mapping γ_1 is linear continuous from $H^2(\Omega)$ to $H^{1/2}(\Gamma)$ (see Prop. 2.4), therefore

(2.9) $$\left|\left\langle\frac{\partial v}{\partial n}, \lambda\right\rangle\right| = |\langle\gamma_1 v,\lambda\rangle| \leq \|\lambda\|_{H^{-1/2}(\Gamma)}\|\gamma_1 v\|_{H^{1/2}(\Gamma)} \leq C\|\lambda\|_{H^{-1/2}(\Gamma)}\|v\|_{H^2(\Omega)}$$

Thus the mapping $v \to \langle \frac{\partial v}{\partial n}, \lambda\rangle$ is continuous from V to R. The conditions of application of the Lax-Milgram theorem being fulfilled, we deduce from it the existence and uniqueness of ψ solution of (2.8).

Let us show that ψ is also the solution of (2.5). The set of $C^\infty(\bar{\Omega})$-functions with compact support, $\mathcal{D}(\Omega)$, being included in V, we have

(2.10) $$\int_\Omega \Delta\psi\Delta v dx = 0 \quad \forall v \in \mathcal{D}(\Omega).$$

Therefore

(2.11) $$\Delta^2\psi = 0.$$

Let $\omega = -\Delta\psi$, then $\omega \in L^2(\Omega)$ and, from (2.11), $\Delta\omega = 0$, therefore

(2.12) $$\omega \in \mathcal{H}.$$

From Green's formula (see N° 2.3) and from (2.11)

$$0 = \int_\Omega \Delta\psi\Delta v dx + \langle\langle\gamma_0 v,\gamma_1\Delta\psi\rangle\rangle - \langle\gamma_1 v,\gamma_0\Delta\psi\rangle \quad \forall v \in H^2(\Omega).$$

If $v \in V$, $\gamma_0 v = 0$, hence

(2.13) $$\int_\Omega \Delta\psi\Delta v dx = \langle\gamma_1 v,\gamma_0\Delta\psi\rangle \quad \forall v \in V.$$

The mapping γ_1 is **surjective** from V onto $H^{1/2}(\Gamma)$ therefore by comparing (2.13) and (2.8) we find that

$$-\gamma_0\Delta\psi = \lambda$$ Q.E.D.

Proof of (ii) : Obviously A is linear. It is also an injection since, from Theorem 2.1, $\psi = 0$ is the unique solution of :

$$\Delta^2\psi = 0 \ , \ \gamma_o\psi = 0 \ , \ \gamma_1\psi = 0.$$

The **surjectivity** of A follows directly from Theorem 2.1 (with $f=0$, $g_1=0$). Therefore A is an algebraic isomorphism of $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

Let us show that A is continuous ; by letting $v = \psi$ in (2.8) we find that

$$\|\Delta\psi\|_{L^2(\Omega)} \le \|\lambda\|_{H^{-1/2}(\Gamma)} \|\gamma_1\psi\|_{H^{1/2}(\Gamma)} \le C\|\lambda\|_{H^{-1/2}(\Gamma)} \|A\psi\|_{L^2(\Omega)},$$

i.e.

$$\|A\psi\|_{L^2(\Omega)} \le C\|\lambda\|_{H^{-1/2}(\Gamma)}.$$

Thus the mapping $A \to \psi : H^{-1/2}(\Gamma) \to V$ is continuous ; then the continuity of A follows from the continuity of $\gamma_1 : H^2(\Omega) \to H^{1/2}(I')$. The continuity of A^{-1} is deduced from the continuity of A by applying the Closed Graph Theorem.

Proof of (iii) : The continuity of A yields the continuity of a(*,*). Let us show the symmetry. Let $\lambda_1,\lambda_2 \in H^{-1/2}(\Gamma)$ and $\psi_1,\psi_2 \in V$ the corresponding solutions of(2.5). From (2.8),

$$(2.14) \qquad \int_\Omega \Delta\psi_1\Delta\psi_2 dx = \langle A\lambda_2,\lambda_1\rangle \quad \forall \lambda_1, A_2 \in H^{-1/2}(\Gamma)$$

and by permuting λ_1 with λ_2

$$\int_\Omega \Delta\psi_2\Delta\psi_1 dx = \langle A\lambda_1,\lambda_2\rangle \quad \forall \lambda_1,\lambda_2 \in H^{-1/2}(\Gamma),$$

which completes the proof of thesynxnetryof a(*,*).

To show the $H^{-1/2}(\Gamma)$-ellipticity, let $\lambda_1 = \lambda_2 = \lambda$ in (2.14) ; then

$$(2.15) \qquad \int_\Omega |\Delta\psi|^2 dx = \langle A\lambda,\lambda\rangle \qquad \forall \lambda \in H^{-1/2}(\Gamma).$$

Since $\Delta\psi \in \mathcal{H}$ (see (2.12)), (2.15), Proposition 2.6 and the fact that $\gamma_o\Delta\psi = -\lambda$, imply that

$$\langle A\lambda,\lambda\rangle \geq C\|\gamma_o\Delta\psi\|^2_{H^{-1/2}(\Gamma)} = C\|\lambda\|^2_{H^{-1/2}(\Gamma)}, \quad C > 0,$$

which completes the proof of Lemma 2.1.■

Let us go back to problem (P_o) with $f \in L^2(\Omega)$, $g_1 \in H^{3/2}(\Gamma)$, $g_2 \in H^{1/2}(\Gamma)$. We have seen from Theorem 2.1. that (P_o) has a unique solution in $H^2(\Omega)$ and that $\omega = -\Delta\psi$ has a trace $\lambda = \gamma_o\omega$ in $H^{-1/2}(\Gamma)$. We shall now show that A is the solution of a linear variational equation in $H^{-1/2}(\Gamma)$.

Let $\bar\psi$ be the unique solution in V of

$$(2.16) \qquad \begin{cases} \Delta^2\bar\psi = 0 \\ \bar\psi|_\Gamma = 0 \\ -\Delta\bar\psi|_\Gamma = \lambda. \end{cases}$$

Problem (2.16) is equivalent to

$$(2.17) \qquad \left| \begin{array}{l} -\Delta\bar\omega = 0 \\ \bar\omega|_\Gamma = \lambda \\ -\Delta\bar\psi = \bar\omega \\ \bar\psi|_\Gamma = 0\,. \end{array} \right.$$

Then let ψ_o be the unique solution in $H^2(\Omega)$ of

$$(2.18) \qquad \begin{cases} \Delta^2\psi_o = f \\ \psi_o|_\Gamma = g_1 \\ -\Delta\psi_o|_\Gamma = 0 \end{cases}$$

which again is equivalent to

$$(2.19)\quad \left|\begin{array}{l} -\Delta\omega_0 = f \\ \omega_o|_\Gamma = 0 \\ -\Delta\psi_0 = \omega_o \\ \psi_o|_\Gamma = g_1. \end{array}\right.$$

Obviously $\psi = \psi_o + \overline{\psi}$, $\omega = \omega_o + \overline{\omega}$. The reader will note that ψ_o is computed by solving two Dirichlet problems for $-\Delta$. Similarly for $\overline{\psi}$, so long as λ is known.
Then one has the following theorem :

Theorem 2.2. : Let ψ be the solution of (P_o) ; then the trace λ of $-\Delta\psi$ on Γ is the unique solution of the linear variational equation

$$(2.20)\quad \begin{cases} \langle A\lambda,\mu\rangle = \langle\frac{\partial\psi_o}{\partial n} - g_2,\mu\rangle \quad \forall\, \mu \in H^{-1/2}(\Gamma), \\ \lambda \in H^{-1/2}(\Gamma). \end{cases}$$

Proof : Prom Lemma 2.1 and from (2.16) we have

$$(2.21)\quad \langle A\lambda,\mu\rangle = -\langle \frac{\partial\overline{\psi}}{\partial n} , \mu\rangle \quad \forall\, \mu \in H^{-1/2}(\Gamma).$$

Since $\overline{\psi} = \psi-\psi_o$ and since $\frac{\partial\psi}{\partial n} = g_2$ on Γ

$$\langle A\lambda,\mu\rangle = \langle\frac{\partial\psi_o}{\partial n} - g_2, \mu\rangle \quad \forall\, \mu \in H^{-1/2}(\Gamma)$$

which shows that λ is a solution of (2.20). The fact that λ is the unique solution of (2.20) follows (via the Lax-Milgram Theorem) from the fact that $\{\lambda,\mu\} \to \langle A\lambda,\mu\rangle$ is bilinear, continuous and $H^{-1/2}(\Gamma)$-elliptic (see Lemma 2.1.) and from the continuity of the linear mapping

$$\mu \to \langle\frac{\partial\psi_o}{\partial n} - g_2, \mu\rangle : H^{-1/2}(\Gamma) \to \mathbb{R}$$

($\frac{\partial\psi_o}{\partial n}$ and g_2 belong to $H^{1/2}(\Gamma)$).■

Remark 2.2. : Since the bilinear form $a(*,*)$ is symmetric, the variational equation (2.20) is equivalent to the minimization problem

$$(2.22)\quad \begin{cases} J(\lambda) \leq J(\mu) \quad \forall \mu \in H^{-1/2}(\Gamma), \\ \lambda \in H^{-1/2}(\Gamma) \end{cases}$$

where

$$J(\mu) = \frac{1}{2} \ \langle A\mu, \mu\rangle - \langle \frac{\partial\psi_o}{\partial n} - g_2, \mu\rangle \ .$$

Remark 2.3 : If the condition : $\frac{\partial\psi}{\partial n}\Big|_\Gamma = g_2$ is treated as a constraint we can associate with (P_o) the Lagrangian $\mathcal{L} : H^2(\Omega) \times H^{-1/2}(\Gamma) \to \mathbb{R}$ defined by

$$(2.23)\qquad \mathcal{L}(v,\mu) = \frac{1}{2} \int_\Omega |\Delta v|^2 dx - \int_\Omega fvdx + \langle\frac{\partial v}{\partial n} - g_2, \ \mu\rangle.$$

Let $\hat{V} = \{v | v \in H^2(\Omega) \ , \ v = g_1 \text{ on } \Gamma\}$; then one could show that $\{\psi, -\gamma_o\Delta\psi\}$ is the unique saddle-point of $\mathcal{L}$ on $\hat{V} \times H^{-1/2}(\Gamma)$ and that

$$J(\mu) = - \underset{v\in\hat{V}}{\text{Min}} \ \mathcal{L}(v,\mu) \ .$$

Therefore (2.22) is the dual problem of (P_o) associated to the Lagrangian $\mathcal{L}$. We refer to [9], [21] for a more complete study of (P_o) by duality methods associated to Lagrangian of the same type of $\mathcal{L}$.

Remark 2.4 : The data f and g_1 come into (2.20) by means of ψ_o only (see (2.18)). •

Remark 2.5 : Let $\tilde{\mu}$ be an extension of μ in Ω. In a formal manner, from Green's formula :

$$(2.24)\quad \begin{cases} a(\lambda,\mu) = \langle A\lambda,\mu\rangle = - \langle\frac{\partial\psi}{\partial n}, \mu\rangle = \\ = - \int_\Omega \Delta\psi\tilde{\mu}dx - \int_\Omega \nabla\psi\cdot\nabla\tilde{\mu}dx = \int_\Omega \omega\tilde{\mu}dx - \int_\Omega \nabla\psi\cdot\nabla\tilde{\mu}dx \end{cases}$$

where ψ is the solution of (2.5) and $\omega = -$ A$. Similarly

$$(2.25)\quad \left\{\begin{aligned} &\langle \frac{\partial\psi_o}{\partial n}, \gamma\,\mu\rangle = \int_\Omega \nabla\psi_o\cdot\nabla\tilde{\mu}dx + \int_\Omega \Delta\psi_o\tilde{\mu}dx = \\ &\qquad = \int_\Omega \nabla\psi_o\cdot\nabla\tilde{\mu}dx - \int_\Omega \omega_o\tilde{\mu}dx \end{aligned}\right.$$

where $\{\omega_o,\psi_o\}$ is the solution of (2.19).

If μ is sufficiently regular (say $\mu \in H^{1/2}(\Gamma)$) so that there exists $\tilde{\mu} \in H^1(\Omega)$ then (2.24) and (2.25) can be justified. The interest of (2.24),(2.25) is that we can now evaluate (2.20) <u>without calculating</u> $\frac{\partial\psi}{\partial n}$ <u>and</u> $\frac{\partial\psi_o}{\partial n}$ <u>explicitly.</u>

We shall take advantage of this remark in Sec. 3 and 4 when (P_o) and (2.20) will be approximated by a <u>mixed finite element method.</u>

2.6. <u>Summary</u>

Let ψ be the solution of

$$(P_o)\quad \left\{\begin{aligned} &\Delta^2\psi = f \text{ in } \Omega \\ &\psi|_\Gamma = g_1 \\ &\frac{\partial\psi}{\partial n}|_\Gamma = g_2 \end{aligned}\right.$$

and $\omega = -\Delta\psi$, $\lambda = \omega|_\Gamma$. We have shown in Sec. 2.5 that for solving (P_o) it is equivalent to solve the following problems :

$$(2.26)\quad \left\{\begin{aligned} &-\Delta\omega_0 = f \text{ in } \Omega, \\ &\omega_o|_\Gamma = 0, \end{aligned}\right.$$

$$(2.27)\quad \left\{\begin{aligned} &-\Delta\psi_o = \omega_o \text{ in } \Omega, \\ &\psi_o|_\Gamma = g_1, \end{aligned}\right.$$

$$(2.28)\quad A\lambda = \frac{\partial\psi_o}{\partial n} - g_2,$$

$$(2.29)\quad \left\{\begin{aligned} &-\Delta\omega = f \text{ in } \Omega, \\ &\omega|_\Gamma = \lambda, \end{aligned}\right.$$

$$(2.30)\quad \left\{\begin{aligned} &-\Delta\psi = \omega \text{ in } \Omega, \\ &\psi|_\Gamma = g_1. \end{aligned}\right.$$

Altogether 4 Dirichlet problems for $-\Delta$ plus an integral equation on Γ whose variational formulation was given in (2.20). In the following sections we shall focus on the approximation of (2.28).

2.7. An explicit example : computation of A when Ω is a disk.

The results of this section are not at all essential for the understanding of the sequel ; they are given for the sake of curiosity. In this section, we assume that

$$\Omega = \{x \in \mathbb{R}^2 \,|\, x_1^2 + x_2^2 < R^2\}.$$

Let (r,θ) be the usual polar coordinate system in $\mathbb{R}^2$.

Theorem 2.3 : Let A be the isomorphism of $H^{-1/2}(\Gamma)$ on $H^{1/2}(\Gamma)$ defined in Sec. 2.5. The eigen functions of A are

$$\text{(2.31)} \qquad \begin{cases} w_{1n}(\theta) = \cos n\theta & n \geq 0 \\ w_{2n}(\theta) = \sin n\theta & n \geq 1, \end{cases}$$

the corresponding eigenvalues being

$$\text{(2.32)} \qquad a_n = \frac{R}{2(n+1)} \quad (n \geq 0).$$

Proof : Let ψ_{1n}, respectively ψ_{2n}, be solutions of

$$\text{(2.33)} \qquad \begin{cases} \Delta^2\psi = 0 \\ \psi|_\Gamma = 0 \\ -\Delta\psi|_\Gamma = \cos n\theta \quad (n \geq 0), \end{cases}$$

$$\text{(2.34)} \qquad \begin{cases} \Delta^2\psi = 0 \\ \psi|_\Gamma = 0 \\ -\Delta\psi|_\Gamma = \sin n\theta \quad (n \geq 1). \end{cases}$$

The reader will check that

$$\text{(2.35)} \qquad \psi_{1n}(r,\theta) = \frac{1}{4(n+1)} \left(\frac{r}{R}\right)^n (R^2-r^2) \cos n\theta \quad \forall\, n \geq 0,$$

(2.36) $$\psi_{2n}(r,\theta) = \frac{1}{4(n+1)} \left(\frac{r}{R}\right)^n (R^2-r^2) \sin n\theta \qquad \forall n \geq 1.$$

Since

$$A\psi = -\frac{\partial\psi}{\partial n}\Big|_\Gamma = -\frac{\partial\psi}{\partial r}(R,\theta)$$

it is seen from (2.35), (2.36) that

$$A\psi_{1n} = \frac{R}{2(n+1)} \cos n\theta \qquad \forall n > 0,$$

$$A\psi_{2n} = \frac{R}{2(n+1)} \sin n\theta \qquad \forall n \geq 1.$$

The sequence $\mathcal{B} = \{w_{10}, w_{11}, w_{21}, \ldots, w_{1n}, w_{2n}, \ldots\}$ = $= \{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta, \ldots\}$ is <u>total in</u> $H^s(\Gamma)$, $\forall s \in \mathbb{R}$ (i.e. the space of linear combinations of elements of $\mathcal{B}$ is <u>dense</u> in $H^s(\Gamma)$) and A is <u>self-adjoint,</u> <u>compact</u> from $L^2(\Gamma)$ into $L^2(\Gamma)$. By applying the spectral theory of self adjoint operator in H-space (see for example RIESZ-NAGY [40]) we conclude that $\mathcal{B}$ is the set of all eigenfunctions of A.

<u>Theorem 2.4</u> : <u>Let</u> A <u>be a sufficiently smooth function on</u> Γ (say $\lambda \in L^2(\Gamma)$) <u>then</u>

$$(A\lambda)(x) = \int_\Gamma A(x,y)\lambda(y)d\Gamma(y) \qquad \forall x \in \Gamma,$$

<u>the kernel of</u> A <u>being</u>

(2.37) $$\begin{cases} A(x,y) = \dfrac{1}{2\pi}\Big[\left(1 - \dfrac{|y-x|^2}{2R^2}\right) \ln\left(\dfrac{R}{|y-x|}\right) + \dfrac{|y-x|}{R}\left(1 - \dfrac{|y-x|^2}{4R^2}\right)^{\frac{1}{2}} \times \\ \times \cos^{-1}\left(\dfrac{|y-x|}{2R}\right) - \dfrac{1}{2}\Big], \end{cases}$$

<u>where</u> $|y-x|$ = <u>distance</u> (x,y).

Proof : In polar coordinates

(2.38) $$(A\lambda)(\theta) = \int_0^{2\pi} A(\theta,\alpha)\lambda(\alpha)d\alpha$$

where

(2.39)
$$A(\theta,\alpha) = \frac{R}{2\pi}\left(\frac{1}{2} + \sum_{n \geq 1} (\cos n\theta \cos n\alpha + \sin n\theta \sin n\alpha) \frac{1}{n+1}\right) =$$
$$= \frac{R}{2\pi}\left(\frac{1}{2} + \sum_{n \geq 1} \frac{\cos n(\theta-\alpha)}{n+1}\right) \qquad \forall \theta \neq \alpha$$

which is the expansion of the kernel $A(\theta,\alpha)$ with respect to the eigenfunctions of A.

Let ϕ = 8-a and $z = e^{i\phi}$; we may assume that $\phi \in]-\pi,+\pi]$; then

(2.40) $$A(\theta,\alpha) = \frac{R}{2\pi} \mathcal{R}e\left(\frac{1}{z} \ln\left(\frac{1}{1-z}\right) - \frac{1}{2}\right) \qquad \forall \theta \neq \alpha.$$

In (2.40) the determination of the complex logarithm is the one which satisfies $\ln 1=0$. Therefore

(2.41) $$A(\theta,\alpha) = -\frac{R}{2\pi}\left(\cos\phi \ln|1-z| + \sin\phi \operatorname{Arg}(1-z) + \frac{1}{2}\right) \quad \forall \phi \neq 0.$$

By inspection of Figure 2.1 we have

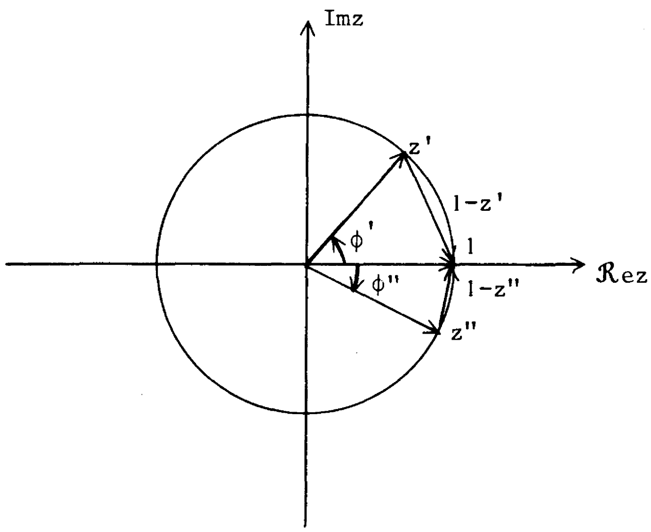

Figure 2.1.

$$(2.42)\qquad \left|\sin\frac{\phi}{2}\right| = \frac{|1-z|}{2} \iff \begin{cases} \phi = 2\sin^{-1}\frac{|1-z|}{2} & \text{if } \phi > 0, \\ \phi = -2\sin^{-1}\frac{|1-z|}{2} & \text{if } \phi < 0. \end{cases}$$

Hence

$$(2.43)\qquad \cos\phi = 1-2\sin^2\frac{\phi}{2} = 1-\frac{|1-z|^2}{2}.$$

But $\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$, therefore

$$(2.44)\qquad \begin{cases} \sin\phi = |1-z|\left(1-\frac{|1-z|^2}{4}\right)^{1/2} & \text{if } \phi > 0, \\ \sin\phi = -|1-z|\left(1-\frac{|1-z|^2}{4}\right)^{1/2} & \text{if } \phi < 0. \end{cases}$$

From Figure 2.1. we also deduce that

$$\mathrm{Arg}\,(1-z) = \frac{\phi}{2} - \frac{\pi}{2} \quad \text{if } \phi > 0,$$

$$\mathrm{Arg}(1-z) = \frac{\phi}{2} + \frac{\pi}{2} \quad \text{if } \phi < 0.$$

Hence from (2.42)

$$(2.45)\qquad \begin{cases} \mathrm{Arg}(1-z) = -\cos^{-1}\frac{|1-z|}{2} & \text{if } \phi > 0, \\ \mathrm{Arg}(1-z) = \cos^{-1}\frac{|1-z|}{2} & \text{if } \phi < 0. \end{cases}$$

Finally, putting back together (2.41),(2.43)-(2.45), if $\theta \neq \alpha$

$$(2.46)\qquad A(\theta,\alpha) = \frac{R}{2}\left[\left(1-\frac{|1-z|^2}{2}\right)\ell n\frac{1}{|1-z|} + |1-z|\left(1-\frac{|1-z|^2}{4}\right)^{1/2} \times \right.$$
$$\left. \times \cos^{-1}\frac{|1-z|}{2} - \frac{1}{2}\right].$$

Let $x = \{R\cos\theta,\ R\sin\theta\}$, $y = \{R\cos a,\ R\sin a)$; (2.38), (2.46) yield

$$(A\lambda)(x) = \frac{1}{2\pi}\int_\Gamma A(x,y)\lambda(y)d\Gamma(y)\ ,$$

where $A(x,y)$ is given by (2.37).■

Remark 2.6 : If the domain Ω is the open disk $(0,R)$ the resolution of the Dirichlet problem on Ω

$$(2.47)\quad \begin{cases} Au = 0 \text{ in } \Omega \text{ (or } \mathbb{R}^2-\Omega) \\ u = g \text{ on } \Gamma \end{cases}$$

involves the operator $B : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ defined by

$$(B\lambda)(x) = \frac{1}{2\pi}\int_\Gamma \ln \frac{R}{|y-x|}\,\lambda(y)\,d\Gamma(y) \qquad \forall\,\lambda \text{ regular}, \ \forall x \in \Gamma.$$

One can show that B is continuous and positive semi-definite, i.e.

$$\langle B\mu,\mu\rangle \geq 0 \qquad \forall\,\mu \in H^{-1/2}(\Gamma).$$

Besides, A and B have the same eigenfunctions (see (2.31)), the corresponding eigenvalues being $\beta_o = 0$, $\beta_n = \frac{R}{2n}$, $n \geq 1$.

For the numerical solution of (2.47) by methods of integral equations on Γ and for more general domains of $\mathbb{R}^2$ and $\mathbb{R}^3$, we refer to NEDELEC-PLANCHARD [35], LEROUX [30] and the bibliography therein.

Remark 2.7 : For more general domains, A will be a pseudo-differential operator, usually not explicitly known. ∎

3. APPROXIMATION OF (P_o) BY A MIXTE FINITE ELEMENT METHOD.

In this section we shall use only polygonal domains Ω, but what follows can easily be extended to the case where isoparametric finite elements (see CIARLET-RAVIART [11]) are used.

3.1. Triangulation of Ω. Fundamental spaces

Let $\mathcal{T}_h$ be a triangulation of Ω satisfying

(3.1) $$\mathcal{T}_h \text{ finite, } T \subset \bar{\Omega} \ \forall T \in \mathcal{T}_h , \quad \bigcup_{T \in \mathcal{T}_h} T = \bar{\Omega} ,$$

(3.2) $$\begin{cases} T \text{ and } T' \in \mathcal{T}_h , \ T \neq T' \Longrightarrow \mathring{T} \cap \mathring{T}' = \emptyset \text{ and } T \cap T' = \emptyset \text{ or } T \text{ and } T' \\ \text{have a side or a vertex in common only,} \end{cases}$$

(3.3) $$h = \text{length of the greatest side of the } T \in \mathcal{T}_h .$$

Let P_k be the space of polynomials of two variables (3 in $\mathbb{R}^3$) of degree less than or equal to k ; we introduce the following finite dimensional spaces

(3.4) $$V_h = \{v_h | v_h \in C^o(\bar{\Omega}) , \ v_h|_T \in P_k \ \forall T \in \mathcal{T}_h\},$$

(3.5) $$V_{oh} = \{v_h | v_h \in V_h , \ v_h = 0 \text{ on } \Gamma\} = V_h \cap H^1_o(\Omega) ,$$

(3.6) $$\begin{cases} \mathcal{M}_h : \text{a complementary space (not precised for the moment)} \\ \text{of } V_{oh} \text{ in } V_h \text{ i.e. } \mathcal{M}_h \subset V_h \text{ and } V_h = V_{oh} \oplus \mathcal{M}_h , \end{cases}$$

(3.7) $$\begin{cases} W_{gh} = \{(v_h,q_h) | (v_h,q_h) \in V_h \times V_h , \ v_h|_\Gamma = g_{1h}, \\ \int_\Omega \nabla v_h \cdot \nabla \mu_h dx = \int_\Omega q_h \mu_h dx + \int_\Gamma g_{2h} \mu_h d\Gamma \quad \forall \mu_h \in V_h\}. \end{cases}$$

In (3.7), g_{1h} is an approximation of g_1 which belongs to $\gamma_o V_h$ and g_{2h} is an approximation of g_2 such that $\int_\Gamma g_{2h} \mu_h d\Gamma$ is "easy" to compute ($g_{2h} = g_2$ possibly).

3.2. Approximation of (P_o)

(P_o) is approximated by

$$(P_h) \quad \left| \begin{array}{l} j_h(\psi_h,\omega_h) \le j_h(v_h,q_h) \ \forall \ (v_h,q_h) \in W_{gh} \ , \\ \\ (\psi_h,\omega_h) \in W_{gh} \end{array} \right.$$

where

$$(3.8) \qquad j_h(v_h,q_h) = \frac{1}{2}\int_\Omega |q_h|^2 dx - \int_\Omega f_h v_h dx \ ,$$

and where f_h is an approximation of f (f_h = f possibly) such that $\int_\Omega f_h v_h dx$ is easy to compute.

Such an approximation (P_h) of (P_o) by finite elements is said to be mixed (see [10],[9]). One can easily show the following proposition :

Proposition 3.1 : Problem (P_h) has one and only one solution.

3.3. Convergence results ($k \ge 2$).

It is assumed that the angles of $\mathcal{T}_h$ are bounded, uniformely in h, from below by $\theta_o > 0$ and that $\mathcal{T}_h$ is such that

$$(3.9) \qquad \underset{T\in\mathcal{T}_h}{\mathrm{Max}}\ h(T) \le \tau \underset{T\in\mathcal{T}_h}{\min}\ h(T) \ , \quad \forall \mathcal{T}_h, \ \tau \text{ independent of } h,$$

where h(T) is the length of the largest side of T. If $k \ge 2$ it is shown in [10] that, under the above hypothesis, one has

$$(3.10) \qquad \|\psi_h-\psi\|_{H^1(\Omega)} + \|\omega_h-(-\Delta\psi)\|_{L^2(\Omega)} \le C\|\psi\|_{H^{k+2}(\Omega)} h^{k-1}$$

where C is independent of h and ψ ; naturally this result supposes that f,g_1,g_2 have been conveniently approximated. For a discussion of the case k=1 we refer to [23,Ch. 4], GLOWINSKI [22]. We also refer to SCHOLZ [43] where, under the above hypothesis on $\mathcal{T}_h$, it is shown that if $k \ge 3$ one has the following error estimate

$$\| \psi_h - \psi \|_{L^2(\Omega)} + h^2 \| \omega_h - (-\Delta\psi) \|_{L^2(\Omega)} \leq C h^{k+1} \|\psi\|_{H^{k+1}(\Omega)}$$

where C is independent of h and ψ.

Remark 3.1. All what is said for triangular elements also holds for quadrangular elements.

3.4. Decomposition of (P_h).

By definition of $\mathcal{M}_h$: $V_h = V_{oh} \oplus \mathcal{M}_h$. Let $\{\psi_h, \omega_h\}$ be the solution of (P_h) and let λ_h be the component of ω_h in $\mathcal{M}_h$, i.e.

(3.11) $$\omega_h = (\omega_h - \lambda_h) + \lambda_h \ ; \ \omega_h - \lambda_h \in V_{oh} \ , \ \lambda_h \in \mathcal{M}_h .$$

In [9] the following theorem is shown

Theorem 3.1 : Let $\{\psi_h, \omega_h\}$ be the solution of (P_h) ; let λ_h be the component of ω_h in $\mathcal{M}_h$; $\{\psi_h, \omega_h, \lambda_h\}$ is also the unique element of $V_h \times V_h \times \mathcal{M}_h$ such that

(3.12) $$\begin{cases} \int_\Omega \nabla\omega_h \cdot \nabla v_h dx = \int_\Omega f_h v_h dx & \forall v_h \in V_{oh} , \\ \omega_h - \lambda_h \in V_{oh}, \end{cases}$$

(3.13) $$\begin{cases} \int_\Omega \nabla\psi_h \cdot \nabla v_h dx = \int_\Omega \omega_h v_h dx & \forall v_h \in V_{oh}, \\ \psi_h \in V_h \ , \ \psi_{h|\Gamma} = g_{1h}, \end{cases}$$

(3.14) $$\int_\Omega \nabla\psi_h \cdot \nabla\mu_h dx = \int_\Omega \omega_h \mu_h dx + \int_\Gamma g_{2h} \mu_h d\Gamma \quad \forall \mu_h \in \mathcal{M}_h .$$

Owing to the importance of this result for what follows we shall give it a proof.

Proof of Theorem 3.1 :

(i) Let W_{oh} be

$$W_{oh} = \{(v_h,q_h) \in V_{oh} \times V_h \mid \int_\Omega \nabla v_h \cdot \nabla \mu_h \, dx = \int_\Omega q_h \mu_h \, dx \quad \forall \mu_h \in V_h\} .$$

Let $\{\psi_h,\omega_h\}$ be the solution of (P_h) then

$$(3.15) \quad \{\psi_h + tv_h, \omega_h + tq_h\} \in W_{gh} \quad \forall t \in \mathbb{R}, \ \forall \{v_h,q_h\} \in W_{oh}.$$

The following process is classical in the Calculus of Variations : from (3.15) we deduce that

$$(3.16) \quad \begin{cases} \frac{1}{t}\left[j_h(\psi_h+tv_h,\omega_h+tq_h) - j_h(\psi_h,\omega_h)\right] \geq 0 \quad \forall t > 0, \\ \forall \{v_h,q_h\} \in W_{oh}. \end{cases}$$

Now

$$\lim_{\substack{t>0 \\ t\to 0}} \frac{1}{t}\left[j_h(\psi_h+tv_h,\omega_h+tq_h) - j_h(\psi_h,\omega_h)\right] = \int_\Omega \omega_h q_h \, dx - \int_\Omega f_h v_h \, dx$$

(then the linear mapping $\{v_h,q_h\} \to \int_\Omega (\omega_h q_h - f_h v_h)\,dx$ is the derivative of j_h at $\{\psi_h,\omega_h\}$).

Therefore

$$\int_\Omega \omega_h q_h \, dx - \int_\Omega f_h v_h \, dx \geq 0 \quad \forall \{v_h,q_h\} \in W_{oh},$$

and since W_{oh} is a linear space,

$$(3.18) \quad \int_\Omega \omega_h q_h \, dx - \int_\Omega f_h v_h \, dx = 0 \quad \forall \{v_h,q_h\} \in W_{oh} .$$

Also by definition of W_{oh}, $\{v_h,q_h\} \in W_{oh}$ implies

$$(3.19) \quad \int_\Omega \nabla v_h \cdot \nabla \omega_h \, dx = \int_\Omega q_h \omega_h \, dx ,$$

which, together with (3.18) implies that

(3.20) $\int_\Omega \nabla\omega_h \cdot \nabla v_h dx = \int_\Omega f_h v_h dx \quad \forall \{v_h, q_h\} \in W_{oh}$.

Let $v_h \in V_{oh}$; the problem

$$\begin{cases} \int_\Omega q_h \mu_h dx = \int_\Omega \nabla v_h \cdot \nabla \mu_h dx & \forall \mu_h \in V_h, \\ q_h \in V_h \end{cases}$$

has a unique solution and of course $\{v_h, q_h\} \in W_{oh}$. This shows that, in (3.20), v_h can be any function of V_{oh} ; therefore it implies (3.12).

Similarly, since $\{\psi_h, \omega_h\} \in W_{gh}$, we have

(3.21) $\int_\Omega \nabla\psi_h \cdot \nabla v_h dx = \int_\Omega \omega_h v_h dx + \int_\Gamma g_{2h} v_h d\Gamma \quad \forall v_h \in V_h$.

Hence (3.13) is proved by choosing $v_h \in V_{oh}$ in (3.21) and v_h in $\mathcal{M}_h$ for (3.14).

(ii) Conversely, since $V_h = V_{oh} \oplus \mathcal{M}_h$, by adding (3.12) and (3.13) we find that

$$\begin{cases} \int_\Omega \nabla\psi_h \cdot \nabla v_h dx = \int_\Omega \omega_h v_h dx + \int_\Gamma g_{2h} v_h d\Gamma & \forall v_h \in V_h, \\ \{\psi_h, \omega_h\} \in V_h \times V_h , \ \psi_{h|\Gamma} = g_{1h}. \end{cases}$$

Therefore

(3.22) $\{\psi_h, \omega_h\} \in W_{gh}$.

Let $\{v_h, q_h\} \in W_{oh}$, then

$$\int_\Omega \nabla v_h \cdot \nabla \mu_h dx = \int_\Omega q_h \mu_h dx \quad \forall \mu_h \in V_h ,$$

and in particular

(3.23) $\int_\Omega \nabla v_h \cdot \nabla \omega_h dx = \int_\Omega q_h \omega_h dx \quad \forall \{v_h, q_h\} \in W_{oh}$.

Then (3.12), (3.22), (3.23) imply

$$(3.24)\quad \begin{cases} \int_\Omega \omega_h q_h dx - \int_\Omega f_h v_h dx = 0 \;\forall \{v_h, q_h\} \in W_{oh}, \\ \{\psi_h, \omega_h\} \in W_{gh}. \end{cases}$$

The functional j_h being convex on $V_h \times V_h$, (3.24) characterizes $\{\psi_h, \omega_h\}$ as being the solution of (P_h).

Remark 3.2 : Equalities (3.12)-(3.14) are the discrete analogues of (2.29),(2.30) and of

$$(3.25)\quad \begin{cases} \int_\Omega \nabla\psi \cdot \nabla\mu dx = \int_\Omega \omega\mu dx + \int_\Gamma g_2 \mu dx \qquad \forall \mu \in \mathcal{M}, \\ \mathcal{M} : \text{complementary of } H_o^1(\Omega) \text{ in } H^1(\Omega). \end{cases}$$

3.5 Discrete analogue of Lemma 2.1.

Let $\lambda_h \in \mathcal{M}_h$ and let ω_h, respectively ψ_h, be the solutions of the following approximate Dirichlet problems

$$(3.26)\quad \begin{cases} \int_\Omega \nabla\omega_h \cdot \nabla v_h dx = 0 \qquad \forall\, v_h \in V_{oh}, \\ \omega_h \in V_h\,,\quad \omega_h - \lambda_h \in V_{oh}\,, \end{cases}$$

$$(3.27)\quad \begin{cases} \int_\Omega \nabla\psi_h \cdot \nabla v_h dx = \int_\Omega \omega_h v_h dx \qquad \forall v_h \in V_{oh}, \\ \psi_h \in V_{oh}. \end{cases}$$

Then we define the bilinear form $a_h : \mathcal{M}_h \times \mathcal{M}_h \to \mathbb{R}$ by

$$(3.28)\qquad a_h(\lambda_h, \mu_h) = \int_\Omega \omega_h \mu_h dx - \int_\Omega \nabla\psi_h \cdot \nabla\mu_h dx \qquad \forall \mu_h \in \mathcal{M}_h.$$

The reader will notice that to define a_h we have used Remark 2.5.

Lemma 3.1. : The bilinear form $a_h(\cdot,\cdot)$ is symetric, positive definite.

Proof : For j=1,2 let $\lambda_{jh} \in \mathcal{M}_h$ and w_{jh}, ψ_{jh} respectively be the solutions of (3.26) and (3.27).

By definition of $a_h(\cdot,\cdot)$ we have

(3.29) $$a_h(\lambda_{1h},\lambda_{2h}) = \int_\Omega \omega_{1h}\lambda_{2h}dx - \int_\Omega \nabla\psi_{1h}\cdot\nabla\lambda_{2h}dx.$$

By letting $\lambda_{2h} = (\lambda_{2h}-\omega_{2h}) + \omega_{2h}$, (3.29) becomes

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(3.30) $$\begin{cases} a_h(\lambda_{1h},\lambda_{2h}) = \int_\Omega \omega_{1h}\omega_{2h}dx - \int_\Omega \nabla\psi_{1h}\cdot\nabla\omega_{2h}dx + \int_\Omega \nabla\psi_{1h}\cdot\nabla(\omega_{2h}-\lambda_{2h})dx \\ - \int_\Omega \omega_{1h}(\omega_{2h}-\lambda_{2h})dx. \end{cases}$$

From (3.26) and since $\psi_{1h} \in V_{oh}$

(3.31) $$\int_\Omega \nabla\omega_{2h}\cdot\nabla\psi_{1h}dx = 0$$

Similarly from (3.27) and since $\omega_{2h}-\lambda_{2h} \in V_{oh}$

(3.32) $$\int_\Omega \nabla\psi_{1h}\cdot\nabla(\omega_{2h}-\lambda_{2h})dx = \int_\Omega w_{1h}(w_{2h}-\lambda_{2h})dx$$

and on account of (3.30)-(3.32) we have

(3.33) $$a_h(\lambda_{1h},\lambda_{2h}) = \int_\Omega \omega_{1h}\omega_{2h}dx \quad \forall\lambda_{jh} \in \mathcal{M}_h \quad j=1,2$$

which shows the symmetry of a_h.

To show the positive definitness let $\lambda_{1h} = \lambda_{2h} = \lambda_h$ in (3.33) then

(3.34) $$a_h(\lambda_h,\lambda_h) = \int_\Omega w_h^2 dx.$$

Therefore $a_h(\lambda_h,\lambda_h) = 0$ implies $\omega_h=0$ which in turn implies $\lambda_h=0$ since λ_h is the component of ω_h in $\mathcal{M}_h$.

3.6. Application of Lemma 3.1. to the resolution of (P_h).
Let $\{\psi_h,\omega_h\}$ be the solution of (P_h) and let λ_h be the component in $\mathcal{M}_h$ of ω_h. Let $\bar\omega_h,\bar\psi_h$ be the solutions of

$$(3.35)\quad \begin{cases} \int_\Omega \nabla\bar\omega_h\cdot\nabla v_h dx = 0 \qquad \forall v_h \in V_{oh}, \\ \bar\omega_h \in V_h \qquad \bar\omega_h-\lambda_h \in V_{oh}, \end{cases}$$

$$(3.36)\quad \begin{cases} \int_\Omega \nabla\bar\psi_h\cdot\nabla v_h dx = \int_\Omega \bar\omega_h v_h dx \quad \forall v_h \in V_{oh}, \\ \bar\psi_h \in V_{oh}. \end{cases}$$

Let ω_{oh} and ψ_{oh} be the solutions of

$$(3.37)\quad \begin{cases} \int_\Omega \nabla\omega_{oh}\cdot\nabla v_h dx = \int_\Omega f_h v_h dx \qquad \forall v_h \in V_{oh}, \\ \omega_{oh} \in V_{oh}, \end{cases}$$

$$(3.38)\quad \begin{cases} \int_\Omega \nabla\psi_{oh}\cdot\nabla v_h dx = \int_\Omega \omega_{oh} v_h dx \quad \forall v_h \in V_{oh}, \\ \psi_{oh} \in V_h \ ,\ \psi_{oh} = g_{1h} \ \text{ on } \Gamma. \end{cases}$$

Then $\psi_h = \bar\psi_h + \psi_{oh}$, $\omega_h = \bar\omega_h + \omega_{oh}$ and (3.35)-(3.38) are the discrete analogues of (2.16)-(2.19).

We shall now show that λ_h is the solution of a variational problem in $\mathcal{M}_h$. From the theorem below we shall derive a discrete analogue of Theorem 2.2.

Theorem 3.2. : Let $\{\psi_h,\omega_h\}$ be the solution of (P_h) and let λ_h be the component in $\mathcal{M}_h$ of w_h. Then λ_h is the unique solution of the linear variational nroblem.

$$(3.39)\quad \begin{cases} a_h(\lambda_h,\mu_h) = \int_\Omega \nabla\psi_{oh}\cdot\nabla\mu_h dx - \int_\Omega \omega_{oh}\mu_h dx - \int_\Gamma g_{2h}\mu_h dx \quad \forall \mu_h \in \mathcal{M}_h, \\ \lambda_h \in \mathcal{M}_h, \end{cases}$$

which is equivalent to a linear system with a positive definite matrix.

Proof : Owing to Lemma 3.1., applied to $\{\bar{\psi}_h,\bar{\omega}_h\}$, we have

$$(3.40)\quad \begin{cases} a_h(\lambda_h,\mu_h) = \int_\Omega \bar{\omega}_h\mu_h dx - \int_\Omega \nabla\bar{\psi}_h\cdot\nabla\mu_h dx = \int_\Omega (\omega_h-\omega_{oh})\mu_h dx - \\ - \int_\Omega \nabla(\psi_h-\psi_{oh})\cdot\nabla\mu_h dx = \int_\Omega \nabla\psi_{oh}\cdot\nabla\mu_h dx - \int_\Omega \omega_{oh}\mu_h dx - \\ - \left(\int_\Omega \nabla\psi_h\cdot\nabla\mu_h dx - \int_\Omega \omega_h\mu_h dx\right) \quad \forall\mu_h \in \mathcal{M}_h, \end{cases}$$

but $\{\psi_h,\omega_h\}$ belongs to W_{gh} therefore (see (3.7))

$$\int_\Omega \nabla\psi_h\cdot\nabla\mu_h dx - \int_\Omega \omega_h\mu_h dx = \int_\Gamma g_{2h}\mu_h dx \quad \forall\ \mu_h \in \mathcal{M}_h$$

which, together with (3.40) proves (3.39).

The uniqueness is obvious since $a_h(\cdot,\cdot)$ is positive definite. The equivalence with a positive definite linear system is a classical result on the approximation of linear variational problems. We shall write the matrix of this system in Section 4.

Remark 3.3 : To compute the right hand side of (3.39) it is necessary to solve two approximate Dirichlet problems ((3.37) and (3.38)). Similarly λ_h being known, to compute w_h and ψ_h it is necessary to solve the two approximate Dirichlet problems (3.12) and (3.13).

3.7. Study of the conditioning of $a_h(\cdot,\cdot)$.

Since the linear system associated with (3.39) will be solved by direct or iterative methods, it is important to know the conditioning of the matrix of the system. Theorem 3.3 below will help to estimate this conditioning. For the sake of clarity we shall assume that Lagrangian type finite elements are used.

Theorem 3.3 : We assume that Ω is convex. If $\mathcal{T}_h$ satisfies the hypothesis of Sec. 3.3., and if $k \geq 1$ and h is sufficiently small, then

$$\alpha h \, \|\gamma_o \lambda_h\|^2_{L^2(\Gamma)} \leq a_h(\lambda_h,\lambda_h) \leq \beta \|\gamma_o \lambda_h\|^2_{L^2(\Gamma)} \quad \forall \lambda_h \in \mathcal{M}_h, \tag{3.41}$$

where α, β are two positive constants, independent of h and λ_h.

Proof :

(i) Proof of the second inequality. Let $\lambda_h \in \mathcal{M}_h$. It follows from (3.34) that

$$a_h(\lambda_h,\lambda_h) = \int_\Omega \omega_h^2 dx, \tag{3.42}$$

where w_h is the solution of

$$\begin{cases} \int_\Omega \nabla\omega_h \cdot \nabla v_h dx = 0 \quad \forall v_h \in V_{oh}, \\ \omega_h - \lambda_h \in V_{oh}. \end{cases} \tag{3.43}$$

Let $\tilde{\omega}_h$ be the solution of the Dirichlet problem

$$\begin{cases} \int_\Omega \nabla\tilde{\omega}_h \cdot \nabla v dx = 0 \quad \forall v \in H^1_o(\Omega), \\ \tilde{\omega}_h - \lambda_h \in H^1_o(\Omega). \end{cases} \tag{3.44}$$

From Sec. 2.5.

$$\begin{cases} \sqrt{a_h(\lambda_h,\lambda_h)} = \|\omega_h\|_{L^2(\Omega)} \leq \|\omega_h - \tilde{\omega}_h\|_{L^2(\Omega)} + \|\tilde{\omega}_h\|_{L^2(\Omega)} = \\ \qquad = \|\omega_h - \tilde{\omega}_h\|_{L^2(\Omega)} + \sqrt{\langle A\gamma_o\lambda_h, \gamma_o\lambda_h \rangle} \; . \end{cases} \tag{3.45}$$

Let $|A| = \|A\|_{\mathcal{L}(L^2(\Gamma), L^2(\Gamma))}$ then (*)

(*) $|A|$ = largest eigenvalue of A.

$$|\langle A\lambda,\mu\rangle| \leq |A| \; \|\lambda\|_{L^2(\Gamma)} \|\mu\|_{L^2(\Gamma)} \qquad \forall \lambda,\mu \in L^2(\Gamma),$$

therefore, from (3.45)

(3.46) $$\sqrt{a_h(\lambda_h,\lambda_h)} \leq \|\tilde{\omega}_h-\omega_h\|_{L^2(\Omega)} + \sqrt{|A|}\; \|\gamma_o\lambda_h\|_{L^2(\Omega)} .$$

To estimate $\|\omega_h-\tilde{\omega}_h\|_{L^2(\Omega)}$ let us use the method of AUBIN-NITSCHE (see [2],[36]).

Let $w \in L^2(\Omega)$ and let ϕ, respectively ϕ_h, be the solutions of

(3.47) $$\begin{cases} \int_\Omega \nabla\phi\cdot\nabla v dx = \int_\Omega wv dx & \forall v \in H_o^1(\Omega) , \\ \phi \in H_o^1(\Omega), \end{cases}$$

(3.48) $$\begin{cases} \int_\Omega \nabla\phi_h\cdot\nabla v_h dx = \int_\Omega wv_h dx & \forall v_h \in V_{oh}, \\ \phi_h \in V_{oh}. \end{cases}$$

Then $-\Delta\phi = v$ and, Ω being convex, $\phi \in H^2(\Omega) \cap H_o^1(\Omega)$. Also $\omega_{h|\Gamma} = \tilde{\omega}_{h|\Gamma} = \lambda_{h|\Gamma}$. Therefore $\omega_h-\tilde{\omega}_h \in H_o^1(\Omega)$. From (3.47) we see that

(3.49) $$\begin{cases} \int_\Omega w(\omega_h-\tilde{\omega}_h)dx = \int_\Omega \nabla\phi\cdot\nabla(\omega_h-\tilde{\omega}_h)dx = \int_\Omega \nabla(\phi-\phi_h)\cdot\nabla(\omega_h-\tilde{\omega}_h)dx + \\ + \int_\Omega \nabla\phi_h\cdot\nabla(\omega_h-\tilde{\omega}_h)dx. \end{cases}$$

Also $\omega_h \in V_{oh} \subset H_o^1(\Omega)$ therefore from (3.43),(3.44), we have

(3.50) $$\int_\Omega \nabla(\omega_h-\tilde{\omega}_h)\cdot\nabla\phi_h dx = 0.$$

Finally from (3.49),(3.50) we have

(3.51) $$\int_\Omega w\,(\omega_h-\tilde{\omega}_h)\,dx = \int_\Omega \nabla(\phi-\phi_h)\cdot\nabla(\omega_h-\tilde{\omega}_h)dx \leq \|\phi_h-\phi\|_{H_0^1(\Omega)} \|\omega_h-\tilde{\omega}_h\|_{H_0^1(\Omega)} .$$

But it is well known (see STRANG-FIX [47]) that under the above hypothesis on $\mathcal{C}_h$,

$$\|\phi_h-\phi\|_{H_o^1(\Omega)} \leq C_1\|\phi\|_{H^2(\Omega)} h$$

where C_1 is a constant independent of h and ϕ.

From Proposition 2.1

$$\|\phi\|_{H^2(\Omega)} \leq C_2 \|w\|_{L^2(\Omega)}.$$

Therefore

$$(3.53) \qquad \int_\Omega w(\omega_h-\tilde{\omega}_h)dx \leq C_3 h\|w\|_{L^2(\Omega)} \|\omega_h-\tilde{\omega}_h\|_{H_o^1(\Omega)} \quad \forall\, w \in L^2(\Omega)$$

which in turn implies that

$$(3.54) \qquad \|\omega_h-\tilde{\omega}_h\|_{L^2(\Omega)} = \sup_{\substack{w \in L^2(\Omega)\\ w \neq 0}} \|w\|^{-1}_{L^2(\Omega)} \left|\int_\Omega w(\omega_h-\tilde{\omega}_h)dx\right| \leq C_3 h \|\omega_h-\tilde{\omega}_h\|_{H_o^1(\Omega)}$$

Thus now we must estimate $\|\omega_h-\tilde{\omega}_h\|_{H_o^1(\Omega)}$; from (3.43),(3.44) we have

$$\int_\Omega \nabla(\omega_h-\tilde{\omega}_h)\cdot\nabla(v_h-\omega_h)dx = 0 \quad \forall v_h \in V_h \;,\; v_h-\lambda_h \in V_{oh}.$$

Therefore

$$(3.55) \qquad \begin{cases} \displaystyle\int_\Omega |\nabla(\omega_h-\tilde{\omega}_h)|^2 dx = \int_\Omega \nabla(\tilde{\omega}_h-\omega_h)\cdot\nabla(\tilde{\omega}_h-v_h)dx \; + \\ \displaystyle + \int_\Omega \nabla(\tilde{\omega}_h-\omega_h)\cdot\nabla(v_h-\omega_h)\, dx = \int_\Omega \nabla(\tilde{\omega}_h-\omega_h)\cdot\nabla(\tilde{\omega}_h-v_h)dx \\ \forall\, v_h \in V_h \;,\; v_h-\lambda_h \in V_{oh} \;, \end{cases}$$

which shows that

$$\|\omega_h-\tilde{\omega}_h\|_{H_o^1(\Omega)} \leq \|v_h-\tilde{\omega}_h\|_{H_o^1(\Omega)} \quad \forall\, v_h \in V_h \;,\; v_h-\lambda_h \in V_{oh}.$$

Let π_h be the operator of interpolation on $\mathcal{T}_h$ associated with the method of finite elements used, $\pi_h \in \mathcal{L}(H^1(\Omega) \cap C^o(\bar{\Omega}), V_h)$. Then $\tilde{\omega}_h \in H^s(\Omega)\ \forall s < \frac{3}{2}$ and

(3.56) $$\|\tilde{\omega}_h\|_{H^s(\Omega)} \leq \Lambda_s \|\gamma_o\lambda_h\|_{H^1(\Gamma)}$$

where Λ_s is independent of λ_h. Now Ω is bounded in $\mathbb{R}^2$ and its boundary is Lipchitz continuous, therefore, $\forall s > 1$, $H^s(\Omega) \subset C^0(\bar{\Omega})$ with continuous injection. Hence π_h can be applied to $\tilde{\omega}_h$ and

(3.57) $$\pi_h\tilde{\omega}_h|\Gamma = \tilde{\omega}_h|\Gamma = \lambda_h|\Gamma$$

Let $s' < s$; owing to the above properties and to the hypothesis on $\mathcal{C}_h$, we have (see for example BABUSKA-AZIZ i 31)

(3.58) $$\|\pi_h\tilde{\omega}_h-\tilde{\omega}_h\|_{H_o^1(\Omega)} \leq C_{(s',s)}h^{s'-1}\|\tilde{\omega}_h\|_{H^s(\Omega)}$$

with $C_{(s',s)}$ independent of h and $\tilde{\omega}_h$. We deduce from (3.56),(3.58)

$$\|\pi_h\tilde{\omega}_h-\tilde{\omega}_h\|_{H_o^1(\Omega)} \leq K_{(s,s')}h^{s'-1}\|\gamma_o\lambda_h\|_{H^1(\Gamma)}$$

with $K_{(s',s)}$ independent of h and λ_h. Therefore $\forall\delta > 0$, there exists C_δ independent of h and λ_h such that

(3.59) $$\|\pi_h\tilde{\omega}_h-\tilde{\omega}_h\|_{H_o^1(\Omega)} \leq C_\delta h^{(1/2-\delta)}\|\gamma_o\lambda_h\|_{H^1(\Gamma)} .$$

From the hypothesis on $\mathcal{C}_h$ we have also

$$\|\gamma_o\lambda_h\|_{H^1(\Gamma)} \leq \frac{C}{h}\|\gamma_o\lambda_h\|_{L^2(\Gamma)} ,$$

therefore

(3.60) $$\|\pi_h\tilde{\omega}_h-\tilde{\omega}_h\|_{H_o^1(\Omega)} \leq C_\delta h^{-1/2-\delta}\|\gamma_o\lambda_h\|_{L^2(\Gamma)}$$

where C_δ is independent of h and λ_h. From (3.57) it is possible to take $v_h = \pi_h\tilde{\omega}_h$ in (3.55) and together with (3.60) it implies that

(3.61) $$\|\omega_h-\tilde{\omega}_h\|_{H_o^1(\Omega)} \leq C_\delta h^{-1/2-\delta}\|\gamma_o\lambda_h\|_{L^2(\Gamma)} ,$$

and at last from (3.46),(3.54)(3.61) we have

$$(3.62)\qquad \sqrt{a_h(\lambda_h,\lambda_h)} \le \left(\sqrt{|A|} + C_\delta h^{1/2-\delta}\right) \|\gamma_o\lambda_h\|_{L^2(\Gamma)} \quad \forall \lambda_h \in \mathcal{M}_h$$

which completes the proof of the second inequality in (3.41).

(ii) Proof of the first inequality.

Since $\mathcal{T}_h$ satisfies the assumptions of Sec. 3.3. it is straightforward to show that

$$(3.63)\qquad \|\gamma_o v_h\|_{L^2(\Gamma)} \le \frac{C}{\sqrt{h}} \|v_h\|_{L^2(\Omega)} \quad \forall v_h \in V_h$$

where C is independent of h and v_h. Recalling that

$$\int_\Omega \omega_h^2 dx = a_h(\lambda_h,\lambda_h) \quad \forall \lambda_h \in \mathcal{M}_h$$

and $\lambda_{h|\Gamma} = \omega_{h|\Gamma}$, we deduce the first inequality of (3.41) immediatly from (3.63). This completes the proof of Theorem 3.3. •

Remark 3.4. : Proceeding as in [9, Th. 10] one could show from Theorem 3.3. that

$$(3.64)\qquad \lim_{h\to 0} \sup_{\lambda_h \in \mathcal{M}_h - \{0\}} \frac{a_h(\lambda_h,\lambda_h)}{\|\gamma_o\lambda_h\|^2_{L^2(\Gamma)}} = \sup_{\lambda \in L^2(\Gamma)-\{0\}} \frac{a(\lambda,\lambda)}{\|\lambda\|^2_{L^2(\Gamma)}} = |A| .$$

3.8. Summary

Let $\{\psi_h,\omega_h\}$ be the solution of (P_h) and let λ_h be the component of ω_h in $\mathcal{M}_h$. The vector λ_h is the solution of a linear system the matrix of which issymmetric, positive definite. This system is given in variational form in (3.39) but the bilinear form $a_h(\cdot,\cdot)$ is not known explicitly. The construction of the matrix and the resolution of the corresponding linear system will be dealt with in the next section. The resolution by iterative schemes will be considered in Section 5.

4. CONSTRUCTION AND RESOLUTION OF THE LINEAR SYSTEM EQUIVALENT TO (3.39).

4.1. Generalities

Let $N_h = \dim(\mathcal{M}_h)$ and $\mathcal{B}_h = \{w_i\}_{i=1}^{N_h}$ a basis for $\mathcal{M}_h$; if $\lambda_h \in \mathcal{M}_h$

(4.1) $$\lambda_h = \sum^{N_h} \lambda_j w_j .$$

Proposition 4.1. : The problem (3.39) is equivalent to the linear system in $(\lambda_1, \ldots \lambda_{N_h})$

$$(E_h) \begin{cases} \displaystyle\sum_{j=1}^{N_h} a_h(w_j, w_i)\lambda_j = \int_\Omega \nabla\psi_{oh} \cdot \nabla w_i \, dx - \int_\Omega \omega_{oh} w_i \, dx - \\ \displaystyle - \int_\Gamma g_{2h} w_i \, d\Gamma, \qquad i = 1, \ldots, N_h. \blacksquare \end{cases}$$

We shall denote $a_{ij} = a_h(w_j, w_i)$, $Ah = (a_{ij})_{i,j=1}^{N_h}$. It is also easy to show the following

Proposition 4.2. : The matrix Ah is a $N_h \times N_h$ positive definite symmetric matrix.■

We shall now study the construction of Ah and of the right member of (E_h) from a suitable basis $\mathcal{B}_h$.

4.2. Choice of $\mathcal{M}_h$.

The space $\mathcal{M}_h$ should be chosen such that the computations of a_{ij} and of the right member of (E_h) are easy. Therefore the basis functions $w_i \in \mathcal{B}_h$ should have a small support. It seems from [9] and [23, Chap. 4] that a good choice is as follows

(4.2) $$\begin{cases} \mathcal{M}_h \text{ complementary of } V_{oh} \text{ in } V_h, \\ v_h \in \mathcal{M}_h \Rightarrow v_h|_T = 0 \ \forall T \in \mathcal{T}_h , \ T \cap \Gamma = \emptyset. \end{cases}$$

If in particular V_h is defined from Lagrangian finite elements (see Figure 4.1. for k=2), $\mathcal{M}_h$ is the space of those functions which take the value zero at all nodes of $\mathcal{T}_h$ which do not belong to Γ .

Then

$$N_h = \dim(\mathcal{M}_h) = \mathrm{Card}\,(\Sigma_h),$$

where

$$\Sigma_h = \{P \in \Gamma \mid P \text{ node of } \mathcal{T}_h\}$$

and a good choice for $\mathcal{B}_h$ is the canonical basis $\mathcal{B}_h = \{w_i\}_{i=1}^{N_h}$, where

$$\text{(4.3)} \quad \begin{cases} w_i \in V_h \\ w_i(P_i) = 1 \,,\ P_i \in \Sigma_h \,,\ w_i(Q) = 0 \ \forall Q \text{ node of } \mathcal{T}_h \,,\ Q \neq P_i . \end{cases}$$

For notational convenience we have supposed that Σ_h has been renumbered from 1 to N_h. With this choice of $\mathcal{M}_h$ and $\mathcal{B}_h$ the coefficients λ_j, in relation (4.1) , of λ_h are precisely the values taken by λ_h at the boundary nodes P_j , $j=1,\dots,N_h$. Thus

$$\lambda_j = \lambda_h(P_j) \qquad \forall P_j \in \Sigma_h \,,\ j=1,\dots,N_h .$$

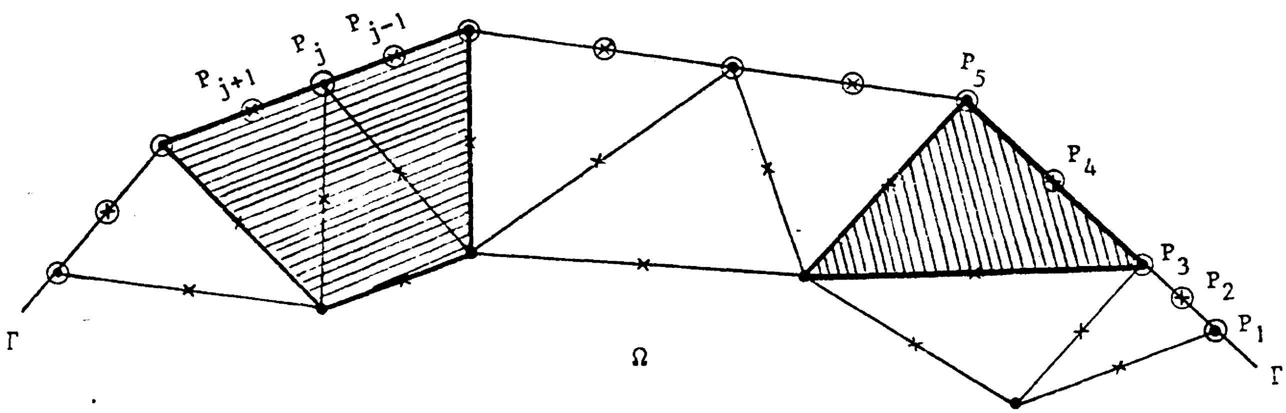

Figure 4.1.

(k=2 ; a small circle indicate a boundary node. The supports of w_4 and w_j are shown).

4.3. Computation of the right members of (E_h)

Let $b_h = \{b_1, \ldots, b_{N_h}\}$ be the vector of the right member of (E_h) :

$$(4.5) \quad b_i = \int_\Omega \nabla\psi_{oh} \cdot \nabla w_i dx - \int_\Omega w_{oh} w_i dx - \int_\Gamma g_{2h} w_i d\Gamma \ , \ i = 1, \ldots, N_h.$$

To compute b_i we need to know ω_{oh} and ψ_{oh}. This is done by solving (3.37), (3.38).

Remark 4.1. : The computation of b_h is faster if the support of w_i is smaller (see Sec. 4.2). Besides if $\mathcal{M}_h$ verifies (4.2) it suffices, to compute b_h, to know ω_{oh} and ψ_{oh} on the triangles $T \in \mathcal{T}_h$ such that $T \cap \Gamma \neq \emptyset$. This remark eventually allows to reduce the memory space allowed to ψ_{oh} and w_{oh} in the computer.

4.4. Computation of the matrix Ah

Let $w_j \in \mathcal{B}_h$. For simplicity let us omit the subscript h on w and ψ. Then let W_j, resp ψ_j, be the solutions of (3.26), resp. (3.27), corresponding to w_j, i.e.

$$(4.6) \quad \begin{cases} \int_\Omega \nabla\omega_j \cdot \nabla\mu_h dx = 0 & \forall \mu_h \in V_{oh}, \\ \omega_j \in V_h & \omega_j - w_j \in V_{oh}, \end{cases}$$

$$(4.7) \quad \begin{cases} \int_\Omega \nabla\psi_j \cdot \nabla\mu_h dx = \int_\Omega \omega_j \mu_h dx & \forall \mu_h \in V_{oh}, \\ \psi_j \in V_{oh}, \end{cases}$$

From (3.17) we find that

$$(4.8) \quad \begin{cases} a_{ij} = a_h(w_j, w_i) = \int_\Omega w_j w_i dx - \int_\Omega \nabla\psi_j \cdot \nabla w_i dx \ , \\ 1 \leq i,j \leq N_h. \end{cases}$$

Thus, to find the j^{th} column of A_h, it is necessary to solve the two approximate Dirichlet problems (4.6)(4.7) and then, w_i describing $\mathcal{B}_h$, to evaluate the integrals in (4.8). Naturally Remark 4.1 also holds for the computation of a_{ij}. It should also be noted that since Ah is symmetric, in the computation of the j^{th} columns, it suffices to compute a_{ij} such that $1 \leq j \leq i$.

We shall see in Sec. 4.5.2. how to use those remarks when (E_h) is solved by the method of CHOLESKY.■

Remark 4.2 : From (3.33) we have

$$a_{ij} = \int_\Omega w_i w_j dx \quad \forall 1 \leq i,j \leq N_h. \tag{4.9}$$

Therefore it seems that A_h can be computed by solving N_h Dirichlet problems, instead of $2N_h$ when (4.8) is used. In fact this simplification is only superficial. Indeed to use (4.9) one needs much more memory for the storage of $w_1, \ldots, \omega_{N_h}$. It is always possible to use tape or disk storage but it increases considerably the computing time. Besides this it should also be noted that the integrals in (4.9) must be calculated over Ω entirely instead of a neighborhood of Γ as in (4.8). ■

4.5. Resolution of (E_h).

4.5.1. Generalities

Let $\overset{\vee}{\lambda}_h \in R^{N_h}$ be the vector $\{\lambda_1, \ldots, \lambda_{N_h}\}$; then (E_h) is written

$$A_h \overset{\vee}{\lambda}_h = b_h. \tag{4.10}$$

The matrix A_h is symmetric positive definite ; to solve (4.10) we can use the method of CHOLESKY. We can also use an iterative method like S.O.R., S.S.O.R. (see VARGA [48], D.M. YOUNG [49]) or like steepest descent or conjugate gradient (see J.W. DANIEL [13], CEA [7], POLAK [39], CONCUS-GOLUB [12]). We shall give more details in Sec. 4.5.2. on CHOLESKY's method which seems particularly well adapted to the resolution of (E_h).

In fact the methods of steepest descent and the conjugate gradient method do not require the knowledge of A_h. We shall come back to this point in Sec. 5.

4.5.2. Resolution of E_h by the method of CHOLESKY.

Since A_h is symmetric positive definite there exists a lower triangular matrix L_h, invertible and unique such that

$$\text{(4.1'1)} \quad \begin{cases} A_h = L_h L_h^t , \\ \ell_{ii} > 0 , \ 1 \leq i \leq N_h , \end{cases}$$

where ℓ_{ii}, $1 \leq i \leq N_h$ are the elements of the diagonal of L_h.

If ℓ_{ij} are the elements of L_h then

$$\ell_{ij} = 0 \text{ if } 1 \leq i < j \leq N_h.$$

We recall the formulae of CHOLESKY :

For $j=1$,

$$\text{(4.12)} \quad \begin{cases} \ell_{11} = \sqrt{a_{11}} , \\ \ell_{i1} = \dfrac{a_{i1}}{\ell_{11}} \quad \forall\, 2 \leq i \leq N_h. \end{cases}$$

For $2 \leq j \leq N_h$

$$\text{(4.13)} \quad \begin{cases} \ell_{jj} = (a_{jj} - \displaystyle\sum_{k=1}^{j-1} \ell_{jk}^2)^{1/2} , \\ \ell_{ij} = \dfrac{1}{\ell_{jj}} (a_{ij} - \displaystyle\sum_{k=1}^{j-1} \ell_{ik}\ell_{jk}) \quad \forall\, j+1 \leq i \leq N_h. \end{cases}$$

It appears from (4.12), (4.13) that it is not necessary to memorize A_h in order to construct L_h. Indeed, suppose that the (j-1)-first columns of L_h are known ; to find the j^{th} column we compute the solution (ω_j,ψ_j) of (4.6),(4.7) and then a_{jj} by (4.8), ℓ_{jj} by (4.13) and a_{ij} by (4.8), ℓ_{ij} by (4.13) for $j+1 \le i \le N_h$. The same argument also applies to the construction of the first column of L_h.

Once L_h is known the determination of λ_h breaks down to the resolution of two triangular systems :

$$(4.14) \qquad \begin{cases} L_h y_h = b_h \\ L_h^t \check{\lambda}_h = y_h . \end{cases}$$

The computation of λ_h from $\check{\lambda}_h$ being straightforward finally ω_h and ψ_h are computed by solving the two approximate Dirichlet problems (3.12),(3.13).

<u>Remark 4.3</u> : Once L_h has been determined it is very easy to solve other problems (E_h) corresponding to other values for f,g_1,g_2. In fact it is a general statement that the most expensive phase of the resolution of a linear system, by CHOLESKY's method is the determination of L_h. It is even more so in our case since the determination of Ah requires the resolution of $2N_h$ Dirichlet problems.

4.5.3. Summary , number of linear sub-problems with the method of CHOLESKY

The solution of (P_h) by (E_h), solved by CHOLESKY's method, requires the resolution of

- Two Dirichlet problems (3.37),(3.38) to compute b_h,
- $2N_h$ Dirichlet problems (4.6),(4.7), $1 \le j \le N_h$, to compute L_h,
- Two linear triangular systems (4.14) to find λ_h,
- Two Dirichlet problems (3.12),(3.13) to compute ω_h, ψ_h.

Thus $2N_h+4$ Dirichlet problems (with the same matrix) and two triangular systems.

4.6. Conditioning of A_h

We recall that the condition number $\nu(M)$ of a square $N \times N$ invertible matrix is given by

$$\nu(M) = \|M\| \, \|M^{-1}\| , \tag{4.15}$$

where the matrix norm is induced by the canonical vector norm of $\mathbb{R}^N$. We recall also that if M is symmetric and positive definite

$$\nu(M) = \frac{\mu_{max}}{\mu_{min}} , \tag{4.16}$$

where μ_{max} (resp. μ_{min}) is the largest (resp. smallest) eigenvalue of M. The linear system (E_h) is easier to solve when $\nu(A_h)$ is small. If $\mathcal{T}_h$ satisfies the assumptions of Sec. 3.3 the following theorem is fairly easy to deduce from Theorem 3.3 and from (4.16).

Theorem 4.1 : If a Lagrangian finite element method is used and if the assumptions in Theorem 3.3 hold, and $k \geq 1$, then

$$\nu(A_h) = O(\frac{1}{h}) . \tag{4.17}$$

Remark 4.4 : It should be pointed out that the classical approximations by finite differences or finite elements of A (resp. Δ^2) lead to matrices with condition number in $O(\frac{1}{h^2})$ (resp. $O(\frac{1}{h^4})$) and are therefore not as well conditioned as A_h.

4.7. Various remarks

Remark 4.5. : The $2N_h+4$ approximate Dirichlet problems found in Sec. 4.5 are of the form

$$(4.18) \qquad (-\Delta)_h u_h = c_h ,$$

where $(-\Delta)_h$ is a $N'_h \times N'_h$ symmetric positive definite matrix (approximating $-\Delta$) with $N'_h = \dim(V_{oh})$.

Therefore since the $2N_h+4$ problems differ only by their right members, the matrix $(-\Delta)_h$ can be factorized by Cholesky's method (and by using the fact that $(-\Delta)_h$ is sparse)

$$(-\Delta)_h = \Lambda_h \Lambda_h^t ,$$

where Λ_h is a lower triangular invertible matrix.
The matrix Λ_h being computed once and for all, the $2N_h+4$ problems reduce to $4N_h+8$ triangular linear systems.

Remark 4.6 : If (4.18) is solved by an iterative method, in order to compute (ω_j, ψ_j), it is not unreasonnable to initialize the algorithm with ω_{j-1}, ψ_{j-1}, provided that the corresponding basis functions w_{j-1}, w_j are neighbors.

Remark 4.7 : All what is said above remains valid if in Sec. 3.1,3.2 numerical integration methods are used to define W_{gh} and (P_h). In particular if $k=1$ and for special triangulations, if $\int_\Omega q_h \mu_h dx$ is approximated by

$$(4.19) \qquad \frac{1}{3} \sum_{T \in \mathcal{C}_h} \text{measure } (T) \sum_{i=1}^{3} q_h(M_{iT})\mu_h(M_{iT}), M_{iT}, i=1,2,3 \text{ vertices of } T,$$

then the method studied gives back the classical 13 points finite difference approximation of the operator Δ^2 (see [23, Ch. 4],[22]).

5. REMARKS ON THE USE OF ITERATIVE METHODS. THE CONJUGATE GRADIENT METHOD.

5.1. Generalities

We have pointed out, already in Sec. 4.5, that (E_h) could be solved by iterative methods such as the method of steepest descent or the conjugate gradient method. We shall see that in doing so it is possible to solve (E_h) without having to compute Ah explicitly. It suffices to solve, at each iteration, two approximate Dirichlet problems for - A.

For the gradient methods we will consider in Sec. 5.2 fixed step size methods, a general study of which was done in CIARLET-GLOWINSKI [9] (see also [23, Ch. 4], [21] and CIARLET [8]) with numerical applications in [5]. However the next paragraph may be viewed as an extention of [9] since iterative schemes in $H^{-1/2}(\Gamma)$, for solving approximatly (P_o), are described. In Sec. 5.3 we shall study some of the methods considered in Sec. 5.2. but with variable steps now. Then in Sec. 5.4 we shall study the conjugate gradient method.

It will be useful for the following to introduce the isomorphism $r_h : \mathcal{M}_h \to \mathbb{R}^{N_h}$ defined below :

Let $\mathcal{B}_h = \{w_i\}_{i=1}^{N_h}$ be the basis of $\mathcal{M}_h$ introduced in Sec. 4.1. If $\mu_h \in \mathcal{M}_h$

(5.1) $$\mu_h = \sum_{i=1}^{N_h} \mu_i w_i ,$$

then r_h is defined by

(5.2) $$r_h\mu_h = \{\mu_1, \mu_2, \dots, \mu_{N_h}\} \quad \forall \mu_h \in \mathcal{M}_h.$$

Let $(\cdot,\cdot)_h$ be the usual euclidian scalar product in $\mathbb{R}^{N_h}$ and $\|\cdot\|_h$ the corresponding norm. Then

(5.3) $$a_h(\lambda_h,\mu_h) = (A_h r_h\lambda_h, r_h\mu_h)_h \quad \forall \lambda_h,\mu_h \in \mathcal{M}_h ,$$

(5.4) $$\int_\Omega \nabla\psi_{oh}\cdot\nabla\mu_h dx - \int_\Omega \omega_{oh}\mu_h dx - \int_\Gamma g_{2h}\mu_h d\Gamma = (b_h, r_h\mu_h)_h \;\forall \mu_h \in \mathcal{M}_h,$$

where A_h and b_h are as in Sec. 4.1, 4.3.

5.2. Steepest descent methods with fixed step size.

5.2.1. Description of the method.

Let $s_h : \mathcal{M}_h \times \mathcal{M}_h \to \mathbb{R}$ be a symmetric positive definite bilinear form and let ρ be a positive number. In a variational formulation the method of steepest descent with fixed step size is written as follow :

(5.5) $$\lambda_h^o \in \mathcal{M}_h \text{ arbitrarily chosen}$$

then λ_h^n known, λ_h^{n+1} is computed by

(5.6) $$\begin{cases} s_h(\lambda_h^{n+1},\mu_h) = s_h(\lambda_h^n,\mu_h)-\rho(a_h(\lambda_h^n,\mu_h)-(b_h,r_h\mu_h)_h) \ \forall\ \mu_h \in \mathcal{M}_h, \\ \lambda_h^{n+1} \in \mathcal{M}_h. \end{cases}$$

Thus to compute λ_h^{n+1} from λ_h^n, it is necessary to solve a variational problem in $\mathcal{M}_h$ i.e. to solve a linear system. We shall come back to this point in Sec. 5.2.2., 5.2.3.

The form s_h being symmetric positive definite there exists a symmetric positive definite matrix S_h such that

(5.7) $$s_h(\lambda_h,\mu_h) = (S_h r_h \lambda_h,\ r_h\mu_h)_h .$$

Now from (5.3),(5.7) we see that (5.5),(5.6) is equivalent to the algorithm :

(5.8) $$r_h\lambda_h^o = \{\lambda_1^o,\dots,\lambda_{N_h}^o\} \in \mathbb{R}^{N_h} \text{ arbitrarily chosen,}$$

(5.9) $$r_h\lambda_h^{n+1} = r_h\lambda_h^n-\rho S_h^{-1}(A_h r_h\lambda_h^n-b_h),$$

which corresponds to more classic notations.

5.2.2. Implementation of algorithm (5.5),(5.6)

In view of (5.9) it appears that to implement (5.5),(5.6) we need to

(i) determinate b_h,

(ii) determinate at each iteration $A_h r_h \lambda_h^n$ (A_h is not known),

(iii) solve a linear system of matrix S_h.

It is seen from (5.4) that the determination of b_h requires the resolution of the two approximate Dirichlet problems (3.37),(3.38) to find ω_{oh} and ψ_{oh}. The implementation of (iii) will be discussed in the next paragraph. As to (ii), Sec. 3.5 and (5.3) imply that to find $A_h r_h \lambda_h^n$ we must solve :

(5.10)
$$\begin{cases} \int_\Omega \nabla\omega_h^n \cdot \nabla v_h dx = 0 & \forall v_h \in V_{oh}, \\ \omega_h^n - \lambda_h^n \in V_{oh} \end{cases}$$

and

(5.11)
$$\begin{cases} \int_\Omega \nabla\psi_h^n \cdot \nabla v_h dx = \int_\Omega \omega_h^n v_h dx & \forall v_h \in V_{oh}, \\ \psi_h^n \in V_{oh} \end{cases}$$

and then $A_h r_h \lambda_h^n$ is such that

$$(A_h r_h \lambda_h^n, r_h \mu_h)_h = a_h(\lambda_h^n, \mu_h) = \int_\Omega \omega_h^n \mu_h dx - \int_\Omega \nabla\psi_h^n \cdot \nabla\mu_h dx \quad \forall \mu_h \in \mathcal{M}_h ;$$

more precisely when μ_h describe $\mathcal{B}_h$ we have

(5.12)
$$(A_h r_h \lambda_h^n)_i = a_h(\lambda_h^n, w_i) \quad \forall i=1,\dots,N_h ,$$

where $(A_h r_h \lambda_h^n)_i$ is the i^{th} component of $A_h r_h \lambda_h^n$ in $\mathbb{R}^{N_h}$.

Once λ_h is obtained (which supposes that the process (5.5),(5.6) converges) ω_h and ψ_h are found by solving (3.12),(3.13).

Remark 5.1. : It is possible to avoid solving the four Dirichlet problems (3.37),(3.38),(3.12),(3.13) by proceeding as follow (as in [5]) :

(5.13) $\lambda_h^o \in \mathcal{M}_h$ arbitrarily chosen,

then λ_h^n known, find λ_h^{n+1} by

$$(5.14)\quad \begin{cases} \int_\Omega \nabla\omega_h^n \cdot \nabla v_h dx = \int_\Omega f_h v_h dx \quad \forall v_h \in V_{oh}, \\ \omega_h^n - \lambda_h^n \in V_{oh}, \end{cases}$$

$$(5.15)\quad \begin{cases} \int_\Omega \nabla\psi_h^n \cdot \nabla v_h dx = \int_\Omega \omega_h^n v_h dx \quad \forall v_h \in V_{oh}, \\ \psi_h^n \in V_h \ , \ \psi_h^n = g_{1h} \text{ on } \Gamma, \end{cases}$$

$$(5.16)\quad \begin{cases} s_h(\lambda_h^{n+1},\mu_h) = s_h(\lambda_h^n,\mu_h)+\rho\left(\int_\Omega \nabla\psi^n \cdot \nabla\mu_h dx - \int_\Omega \omega_h^n \mu_h dx - \int_\Gamma g_{2h}\mu_h d\Gamma\right), \\ \forall \mu_h \in \mathcal{M}_h \ , \ \lambda_h^{n+1} \in \mathcal{M}_h. \end{cases}$$

In view of (5.7),(5.16) the determination of λ_h^{n+1} in (5.16) requires the resolution of a linear system of matrix S_h.

5.2.3. Choice of s_h

In principle any symmetric positive definite bilinear form on $\mathcal{M}_h$ will work. However the choice of s_h should be guided by the following two seemingly contradictory properties.

(i) Choose $s_h(\cdot,\cdot)$ such that S_h is sparse and even diagonal ; in the former case S_h will be factorized by the Cholesky method, $S_h = T_h T_h^t$, and T_h will be stored in the memory of the computer (T_h is also sparse).

(ii) Since $a_h(\cdot,\cdot)$ is an approximation of $a(*,*)$ defined over $H^{-1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$-elliptic, it would seem reasonnable to take $s_h(\cdot,\cdot)$ to be also an approximation of $s(\cdot,\cdot)$ where $s(\cdot,\cdot)$ is a bilinear form, $H^{-1/2}(\Gamma)$-elliptic. Such a choice leads to a full matrix S_h ; we shall come back to this point in Sec. 5.2.5.(See also Remark 8.1).

Let us discuss the point of view (i) : let us assume that $\mathcal{M}_h$ is defined by (4.2) and that a Lagrangian finite element method is used. It follows from [9], [23, Ch. 4], [5] that $s_h(\cdot,\cdot)$ can be one of the following

$$s_h(\lambda_h,\mu_h) = \int_\Gamma \lambda_h\mu_h d\Gamma , \tag{5.17}$$

$$s_h(\lambda_h,\mu_h) = \int_\Omega \lambda_h\mu_h dx \tag{5.18}$$

$$s_h(\lambda_h,\mu_h) = \int_\Omega \nabla\lambda_h\cdot\nabla\mu_h dx. \tag{5.19}$$

Such choices lead to a sparse matrix S_h(provided that the boundary nodes have been properly numbered).

By numerical integration it is easy to approximate (5.17),(5.18) by bilinear forms for which S_h is diagonal. If k=1 (resp k=2) and if the notations are as in Figure 5.1. (resp. 5.2) we may approximate (5.17) by

$$s_h(\lambda_h,\mu_h) = \sum_{i=1}^{N_h} \frac{|P_{i-1}P_i|+|P_iP_{i+1}|}{2} \lambda_1\mu_1 , \tag{5.20}$$

which corresponds to the trapezoidal rule of integration (resp. by

$$s_h(\lambda_h,\mu_h) = \sum_{i=1}^{N_h} \frac{|P_iP_{i+1}|}{6} (\lambda_i\mu_i+4\lambda_{i+\frac{1}{2}}\mu_{i+\frac{1}{2}}+\lambda_{i+1}\mu_{i+1}) , \tag{5.21}$$

which corresponds to Simpson's rule).

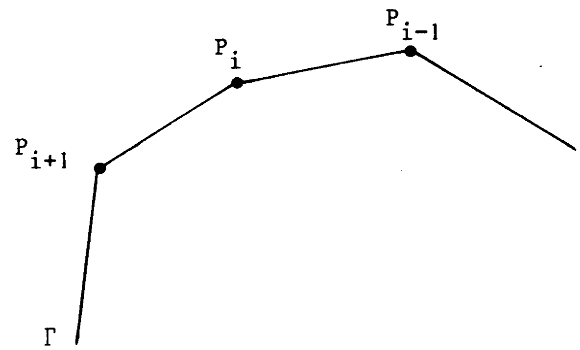

Figure 5.1.

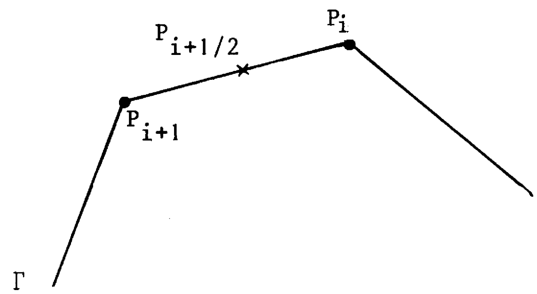

Figure 5.2.

5.2.4. Convergence of algorithm (5.5),(5.6)

Theorem 5.1. : Let $\{\lambda_h^n\}_n$ be a sequence generated by algorithm (5.5),(5.6) and λ_h the solution of (E_h). Then for all choices $\lambda_h^o \in \mathcal{M}_h$

$$\lim_{n\to\infty} \lambda_h^n = \lambda_h ,$$

if and onlv if

(5.22) $$0 < \rho < \frac{2}{\Lambda_{N_h}}$$

where Λ_{N_h} is the largest eigenvalue of $S_h^{-1}A_h$.

Proof : Let $y_h^n = r_h\lambda_h^n - r_h\lambda_h$. From (5.9)

(5.23) $$y_h^{n+1} = \quad (I - + \& \quad .$$

The matrices A_h and S_h being symmetric positive definite, $S_h^{-1}A_h$ has N_h eingenvalues

$$0 < \Lambda_1 \leq \Lambda_2 \leq ,\dots, \leq \Lambda_{N_h}$$

and N_h eigenvectors $\{v_i\}_{i=1}^{N_h}$ S_h-orthogonal i.e.

$$(S_h v_i, v_j)_h = 0 \qquad i \neq j \quad ;$$

the set $\{v_i\}_i^{N_h}$ being a basis for R^{N_h}, y_h^n can be computed on it and with self explanatory notations, (5.23) becomes

(5.24) $$y_i^{n+1} = (1-\rho\Lambda_i)y_i^n \quad i=1,\dots,N_h.$$

Algorithm (5.5),(5.6) will converge if and only if

(5.25) $$|1-\rho\Lambda_i| < 1 \qquad \forall i=1,\dots,N_h \ ,$$

which is equivalent to (5.22). ∎

Remark 5.2 : One could show that

(5.26) $$\Lambda_{N_h} = \max_{\mu_h \in \mathcal{M}_h - \{0\}} \frac{a_h(\mu_h,\mu_h)}{s_h(\mu_h,\mu_h)} .$$

Remark 5.3 : The previous demonstration, based on the spectral decomposition of $S_h^{-1}A_h$ is standard. Another method, based on inequalities of energy can be found in [9] ; this method extends to the infinite dimension case (see [9, Sec. 21).

Remark 5.4 : If $s_h(\lambda_h,\mu_h) = \int_\Gamma \lambda_h \mu_h d\Gamma$ (which is the most natural choice for s_h) it is shown in [9] that under the hypothesis on $\mathcal{T}_h$ in Theorem 3.3 and for Lagrangian finite elements with $k \geq 1$, (5.27) $$\lim_{h\to 0} \Lambda_{N_h} = \max_{\substack{v \in H^2(\Omega) \cap H^1_o(\Omega) \\ v \neq 0}} \{ \int_\Gamma |\frac{\partial v}{\partial n}|^2 d\Gamma \Big/ \int_\Omega |\Delta v|^2 dx \} = A ,$$

therefore it is possible to estimate Λ_{N_h} for a certain number of domains for which A is known (see J. SMITH [44]).

Remark 5.5 : It can be shown from (5.24) that the optimal value for ρ is

(5.28) $$\rho_{opt} = 2/(\Lambda_1+\Lambda_{N_h}) ,$$

in which case

(5.29) $$|y_i^{n+1}| \leq \frac{\Lambda_{N_h}-\Lambda_1}{\Lambda_{N_h}+\Lambda_1} |y_i^n| \quad \forall\, i=1,\dots,N_h ,$$

which gives a linear convergence ratio

(5.30) $$\mathcal{R}_{opt} \leq \frac{\Lambda_{N_h}-A_1}{\Lambda_{N_h}+\Lambda_1} .$$

With (5.17) and according to Theorems 3.3 and 4.1,

(5.31) $$\mathcal{R}_{opt} \leq 1-\gamma h \; , \; \gamma > 0 \text{ independent of } h.$$

This result seems pessimistic at first sight. However numerical tests show that if the solution of (P_o) is smooth the speed of convergence is practically independent of h, (see [5]). This is because algorithm (5.5),(5.6) is a finite dimensional approximation of the continuous algorithm below

(5.32) $$\lambda^o \in L^2(\Gamma) \text{ arbitrarily chosen,}$$

(5.33) $$\begin{cases} -\Delta\omega^n = f, \\ \omega^n|_\Gamma = \lambda^n, \end{cases}$$

(5.34) $$\begin{cases} -\ \Delta\psi^n = \omega^n , \\ \psi^n|_\Gamma = g_1, \end{cases}$$

(5.35) $$\lambda^{n+1} = \lambda^n + \rho\left(\frac{\partial\psi^n}{\partial n} - g_2\right) .$$

Let ψ be the solution of (P_0) ; if $\lambda = -\Delta\psi|_\Gamma = \omega|_\Gamma$ belongs to $L^2(\Gamma)$ it follows from [9],[22], provided

(5.36) $$0 < \rho < \frac{2}{\Lambda} \qquad (\Lambda \text{ defined in } (5.27)),$$

'that

(5.37) $$\lim_{n\to\infty} \{\psi^n,\omega^n\} \to \{\psi,-\Delta\psi\} \text{ in } H^2(\Omega) \times L^2(\Omega), \text{ strongly.}$$

However one can show that in general the convergence rate is sub-linear (i.e. slower than any geometric sequence). This is due to the fact that A, introduced in Sec. 2.5, is compact from $L^2(\Gamma)$ into $L^2(\Gamma)$.

Now let us construct steepest descent methods in $H^{-1/2}(\Gamma)$.

5.2.5. Iterative methods in $H^{-1/2}(\Gamma)$.

In this section we assume that Ω is simply connected. Let us investigate the point of view (ii) of Sec. 5.2.3. Among the continuous bilinear forms $s(\cdot,\cdot)$, $H^{-1/2}(\Gamma)$-elliptic, the most classical one (see Remark 2.6) is defined by

(5.38) $$s(\lambda,\mu) = \frac{1}{2\pi}\int_{\Gamma\times\Gamma} \left(C+\ell n \frac{1}{|y-x|}\right)\lambda(x)\mu(y)d\Gamma(x)d\Gamma(y)$$

where C is a positive constant.

Of course if a Lagrangian finite element method is used and if $\mathcal{M}_h$ is defined by (4.2) then $s(\cdot,\cdot)$ defines a symmetric positive definite form $s_h(\cdot,\cdot)$ by

$$(5.39)\quad \begin{cases} s_h(\lambda_h,\mu_h) = s(\gamma_o\lambda_h,\gamma_o\mu_h) = \frac{1}{2\pi}\int_{\Gamma\times\Gamma}(C+\ell n\,\frac{1}{|y-x|})\lambda_h(x)\mu_h(y)\,d\Gamma(x)\,d\Gamma(y), \\ \lambda_h,\mu_h \in \mathcal{M}_h\,, \end{cases}$$

and we recall that γ_o is the trace mapping of Sec. 2.2. In practice $s_h(\cdot,\bullet)$ of (5.39) is not feasible and we must approximate the integral in the right member by a numerical integration process (see LEROUX [30]). Then a (full) matrix S_h is obtained and, once factorized, $S_h = T_hT_h^t$, the matrix T_h will be stored in the memory of the computer. However we prefer the following process, which ought to be justified theoretically. For clarity **we** assume **k=1** and we start with the following remark :

If $\hat{\gamma}$ is the **cercle** of radius $\frac{1}{2\pi}$ and centre 0 (see Figure 5.3.)

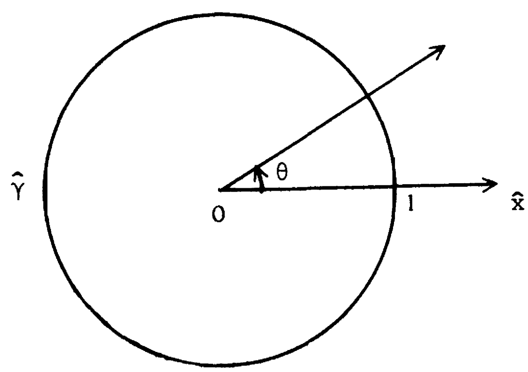

Figure 5.3.

then the operator

(5.40) $$\hat{S} = (I - \frac{d^2}{d\theta^2})^{-1/2}$$

is an isomorphism of $H^{-1/2}(\hat{\gamma})$ on $H^{1/2}(\hat{\gamma})$, symmetric and $H^{-1/2}(\hat{\gamma})$-elliptic. Now let us approximate S ($\hat{S}^{-1}$, in fact) as follows : let $\hat{h} = 1/N_h$ and define $\Delta_h : R^{N_h} \to \mathbb{R}^{N_h}$ by

(5.41) $$\begin{cases} (\hat{\Delta}_h \xi)_1 = (\xi_2 + \xi_{N_h} - 2\xi_1)/\hat{h}^2 \\ (\hat{\Delta}_h \xi)_i = (\xi_{i+1} + \xi_{i-1} - 2\xi_i)/\hat{h}^2 \ , \ 2 \leq i \leq N_h - 1, \\ (\hat{\Delta}_h \xi)_{N_h} = (\xi_1 + \xi_{N_h - 1} - 2\xi_{N_h})/\hat{h}^2 \ , \ \forall\, \xi \in \mathbb{R}^{N_h}. \end{cases}$$

The operator $-\hat{\Delta}_h$ (resp. $I-\hat{\Delta}_h$) is symmetric positive semi-definite (resp. positive definite) and in **matricial** form,

(5.42) $$-\hat{\Delta}_h = \frac{1}{\hat{h}^2}\begin{pmatrix} 2 & -1 & & & & & -1 \\ -1 & 2 & -1 & & & 0 & \\ & \ddots & \ddots & \ddots & & & \\ & & -1 & 2 & -1 & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & -1 & 2 & -1 \\ -1 & & & & & -1 & 2 \end{pmatrix}$$

The interest of $-\hat{\Delta}_h$ is in the fact that its eigenvalues and eigenvectors are known explicitly , therefore the computation of

(5.43) $$S_h^{-1} = (I - \hat{\Delta}_h)^{1/2}$$

will be easy. In fact the reader will check that if N_h is even the eigenvectors of $-\hat{\Delta}_h$ are

$$(5.44)\quad \begin{cases} C_j = \{\cos 2\pi j(i-1)\hat{h}\}_{i=1}^{N_h}, & 0 \le j \le \frac{N_h}{2}, \\ s_j = \{\sin 2\pi j(i-1)\hat{h}\}_{i=1}^{N_h}, & 1 \le j \le \frac{N_h}{2} - 1, \end{cases}$$

and the corresponding eigenvalues are

$$(5.45)\quad \delta_j = \frac{4}{\hat{h}^2} \sin^2 \frac{\pi j \hat{h}}{2}.$$

If N_h is odd then

$$(5.46)\quad \begin{cases} C_j = \{\cos 2\pi j(i-1)\hat{h}\}_{i=1}^{N_h}, & 0 \le j \le \frac{N_h - 1}{2}, \\ s_j = \{\sin 2\pi j(i-1)h\}_{i=1}^{N_h}, & 1 \le j \le \frac{N_h - 1}{2}, \end{cases}$$

with the eigenvalues as in (5.45). Then to compute S_h^{-1} we normalize C_j and S_j

$$(5.47)\quad \begin{cases} \bar{C}_j = C_j / \|C_j\|_h, \\ \bar{S}_j = S_j / \|S_j\|_h, \end{cases}$$

and we denote by T_h the unitary matrix that has $\bar{C}_j$ and $\bar{S}_j$ as column vectors

$$(5.48)\quad T_h = (\bar{C}_o \bar{C}_1 \bar{S}_1 \ldots \bar{C}_j \bar{S}_j \ldots).$$

Then we denote by

$$(5.49)\quad D_h = \begin{pmatrix} 1 & & & & & & \\ & 1+\delta_1 & & & & \bigcirc & \\ & & 1+\delta_1 & & & & \\ & & & \ddots & & & \\ & & & & 1+\delta_j & & \\ & \bigcirc & & & & 1+\delta_j & \\ & & & & & & \ddots \end{pmatrix},$$

and then

$$I-\hat{\Delta}_h = T_h D_h T_h^t$$

and

$$S_h^{-1} = (I-\hat{\Delta}_h)^{1/2} = T_h D_h^{1/2} T_h^t \ , \qquad (5.50)$$

and of course

$$D_h^{1/2} = \begin{pmatrix} \sqrt{1+\delta_1} & & & & 0 \\ & \ddots & & & \\ & & \sqrt{1+\delta_j} & & \\ & & & \ddots & \\ 0 & & & & \sqrt{1+\delta_{N_h}} \end{pmatrix} .$$

The matrices S_h and S_h^{-1} are full $N_h \times N_h$ symmetric positive definite matrices.

Algorithm (5.5),(5.6) (in its equivalent form (5.8)(5.9)) has been applied to (P_o) with S_h defined by (5.50) and the numbering of Γ being as in Figure 5.1. The corresponding numerical experiments will be described in a forthcoming publication by BOURGAT-GLOWINSKI-PIRONNEAU. In Sec. 8, Remark 8.1 we suggest an alternate choice for $s_h(\cdot,\cdot)$, in order to iterate "approximately" in $H^{-1/2}(\Gamma)$.

5.3. Gradient method with variable step size.

5.3.1. Orientation

Fixed step size steepest descent method have the drawback to require the knowledge of the eigenvalues Λ_1 and Λ_{N_h} to find a feasible ρ. At the cost of additional computations one may overcome this difficulty. We shall indicate two methods for adjusting ρ at each iteration and we shall give some details on the implementation of these methods. These two methods are well-known as steepest descent method and minimum residual method.

5.3.2. Principle of the variable step size methods.

Let us first begin by recalling the principles of these methods and then in the subsequent sections their applications to (E_h).
In $\mathbb{R}^N$, let $\mathcal{A}$ be a $N\times N$ symmetric positive definite matrix and $\beta \in \mathbb{R}^N$. The linear system

(5.51) $$\mathcal{A}\xi = \beta$$

has a unique solution. Let us solve (5.51) by the following algorithm

(5.52) $$\xi^o \in \mathbb{R}^N \text{ arbitrarily chosen ,}$$

(5.53) $$\xi^{n+1} = \xi^n - \rho_n S^{-1} (A\, \xi^n - \beta) ,$$

where in (5.53) S is a $N\times N$ symmetric positive definite matrix and ρ_n is chosen "at best" at each iteration. We denote

(5.54) $$g_n = \mathcal{A}\xi^n - \beta \ .$$

(1) Method of steepest descent.

Let $J : \mathbb{R}^N \to R$ defined by

$$J(\eta) = \frac{1}{2} (\mathcal{A}\eta,\eta) - (\beta,\eta) \ .$$

Then the solution of (5.51) is the unique solution of the minimization problem

$$\begin{cases} J(\xi) \le J(\eta) \quad \forall\, \eta \in \mathbb{R}^N , \\ \xi \in \mathbb{R}^N. \end{cases}$$

Therefore ρ_n is computed such that

(5.55) $$\begin{cases} J(\xi^n - \rho_n S^{-1} g^n) \le J(\xi^n - \rho S^{-1} g^n) \ \forall\, \rho \in \mathbb{R} , \\ \rho_n \in \mathbb{R}. \end{cases}$$

It follows from (5.55) that

$$\rho_n = (S^{-1}g^n, g^n)/(\mathcal{A}S^{-1}g^n, S^{-1}g^n) \tag{5.56}$$

which completes (5.52)(5.53). Let us note that g^n satisfies

$$g^{n+1} = g^n - \rho_n \mathcal{A}S^{-1}g^n. \tag{5.57}$$

This relation will play an important role in the resolution of (E_h).

(2) Method of minimum residual.

We have still (5.52),(5.53), but ρ_n is such that

$$\begin{cases} (S^{-1}g^{n+1}, g^{n+1}) \le (S^{-1}(g^n - \rho\mathcal{A}S^{-1}g^n), g^n - \rho\mathcal{A}S^{-1}g^n) \quad \forall \rho \in \mathbb{R}, \\ \rho_n \in \mathbb{R}, \end{cases} \tag{5.58}$$

from which we find

$$\rho_n = (\mathcal{A}S^{-1}g^n, S^{-1}g^n)/(S^{-1}\mathcal{A}S^{-1}g^n, \mathcal{A}S^{-1}g^n). \tag{5.59}$$

The relation (5.57) still holds in this case.

5.3.3. Application of the method of steepest descent for the solution of (E_h).

In the particular case of (E_h), algorithm (5.52),(5.53),(5.56) takes the form :

$$\lambda_h^o \in \mathcal{M}_h, \tag{5.60}$$

$$r_h\lambda_h^{n+1} = r_h\lambda_h^n - \rho_n S_h^{-1} g_h^n, \tag{5.61}$$

$$\rho_n = (S_h^{-1}g_h^n, g_h^n)_h/(A_hS_h^{-1}g_h^n, S_h^{-1}g_h^n)_h, \tag{5.62}$$

$$g_h^n = A_h r_h \lambda_h^n - b_h. \tag{5.63}$$

In case Ah is known explicitly the implementation of (5.60)-(5.63) is straightforward. Besides we think that it is not interesting to use this method for solving (E_h) when Ah is known explicitly. Therefore let us assume that Ah has not been computed yet. By inspection of (5.61),(5.63) it seems that two Dirichlet problems must be solved at each iteration to compute g_h^n, then a linear system of matrix S_h to find $S_h^{-1}g_h^n$ and again two Dirichlet problems for $A_hS_h^{-1}g_h^n$. However from (5.57)

(5.64) $$g_h^{n+1} = g_h^n-\rho_n A_h S_h^{-1} g_h^n$$

so that one may proceed like this :

- Compute g_h^o from λ_h^o (two Dirichlet problems) and compute $S_h^{-1}g_h^o$ and $A_hS_h^{-1}g_h^o$ (two more Dirichlet problems) then compute ρ_o by (5.62) and $\lambda_h^1, g^1{}_h$ by (5.61),(5.64).

- Compute $\lambda_h^{n+1}, g_h^{n+1}, \rho_n$ from g_h^n by computing $S_h^{-1}g_h^n$ and $A_hS_h^{-1}g_h^n$(two Dirichlet problems) and by using (5.61),(5.64),(5.62).

<u>In short</u> :

- One needs at each iteration to solve a linear system of matrix S_h,
- And two Dirichlet problems per iteration (+ two more for the first iteration).

This procedure is summarized as follows :

(5.65) $$\lambda_h^o \in \mathcal{M}_h,$$

(5.66) $$g_h^o = A_h r_h \lambda_h^o - b_h \ , \ n=0 \ ,$$

(5.67) $$\rho_n = (S_h^{-1}g_h^n, g_h^n)_h / (A_hS_h^{-1}g_h^n, S_h^{-1}g_h^n)_h \ ,$$

(5.68) $$r_h\lambda_h^{n+1} = r_h\lambda_h^n - \rho_n S_h^{-1} g_h^n,$$

(5.69) $$g_h^{n+1} = g_h^n - \rho_n A_h S_h^{-1} g_h^n \ ,$$

n=n+1 and go to (5.67).

Remark 5.6. : We could study the rate of convergence of algorithm (5.60)-(5.63) by the techniques developped in MARCHOUK-KUZNETSOV [33]. Similarly it would be interesting to study the propagation of round-off errors in the numerical implementation of (5.65)-(5.69).

Remark 5.7. : When the Dirichlet problems and the linear system of matrix S_h are solved by direct methods the feasibility of a pre-factorization method (like Cholesky for instance) is evident.

5.3.4. Implementation of the minimum residual method for solving (E_h)

Everything said in Sec. 5.3.3. for the steepest descent method applies also for the minimum residual algorithm. The two methods differ only by their choices of ρ_n. The adaptation will be obtained by replacing in algorithm (5.60)-(5.63) instruction (5.62) by

(5.70) $$\rho_n = (A_h S_h^{-1} g_h^n, S_h^{-1} g_h^n)_h / (S_h^{-1} A_h S_h^{-1} g_h^n, A_h S_h^{-1} g_h^n)_h.$$

Similarly when Ah is not known explicitly it is better to use (5.65)-(5.69) with (5.67) replaced by (5.70). Remarks 5.6, 5.7 also apply to this algorithm.

5.4. Solution of (E_h) by the conjugate gradient method

5.4.1. Orientation.

The matrix Ah being symmetric positive definite it is natural to solve (E_h) by a conjugate gradient method. We recall that these methods are super-linearly convergent and that when there are no round-off errors they converge in a finite number of iterations.

We begin, in Sec. 5.4.2., by some recalls on the conjugate gradient method and then in Sec. 5.4.3. its implementation for solving (E_h) is discussed.

5.4.2. Recalls on the conjugate gradient method.

Again let us consider problem (5.51), i.e. $\mathcal{A}\xi = \beta$, where $\mathcal{A}$ satisfies the hypothesis of Sec. 5.3.2. For this problem the conjugate gradient method is (see for example [13], [71, [39]).

(5.71) $\xi^0 \in \mathbb{R}^N$, chosen arbitrarily,

(5.72) $g^0 = \mathcal{A}\xi^0 - \beta$,

(5.73) $z^0 = g^0$, n=0 ,

(5.74) $\rho_n = (z^n, g^n) / (\mathcal{A}z^n, z^n)$ (*)

(5.75) $\xi^{n+1} = \xi^n - \rho_n z^n$

(5.76) $\gamma_n = (g^{n+1}, g^{n+1}) / (g^n, g^n)$

(5.77) $z^{n+1} = g^{n+1} + \gamma_n z^n$

n=n+1 and go to (5.74).

Note that (5.75) implies that

(5.78) $g^{n+1} = g^n - \rho_n \mathcal{A}z^n$.

This relation will play an important role in the resolution of (E_h).

5.4.3. Implementation on (E_h)

In the particular case of (E_h), (5.71)-(5.77) takes the form

(5.79) $\lambda_h^0 \in \mathcal{M}_h$, chosen arbitrarily,

(5.80) $g_h^0 = A_h r_h \lambda_h^0 - b_h$

(*) We also have $\rho_n = |g^n|^2 / (\mathcal{A}z^n, z^n)$.

(5.81) $z_h^o = g_h^o$, $n=0$,

(5.82) $\rho_n = (z_h^n, g_h^n)_h / (A_h z_h^n, z_h^n)_h$ (or $(g_h^n, g_h^n)_h / (A_h z_h^n, z_h^n)_h$) ;

(5.83) $r_h \lambda_h^{n+1} = r_h \lambda_h^n - \rho_n z_h^n$,

(5.84) $g_h^{n+1} = A_h r_h \lambda_h^{n+1} - b_h$,

(5.85) $\gamma_n = (g_h^{n+1}, g_h^{n+1})_h / (g_h^n, g_h^n)_h$

(5.86) $z_h^{n+1} = g_h^{n+1} + \gamma_n z_h^n$,

$n=n+1$, go to (5.82) .■

By inspection of (5.82),(5.84),(5.85) it seems that 4 Dirichlet problems are required at each iteration to implement (5.79)-(5.86) (two for $A_h z_h^n$ (resp. $A_h r_h \lambda_h^{n+1}$)). In fact as for algorithms of Sec. 5.3. one can reduce the number of Dirichlet problems to two. Because

$(5.84)_{bis}$ $g_h^{n+1} = g_h^n - \rho_n A_h z_h^n$.

Indeed if we use algorithm (5.79)-(5.83),$(5.84)_{bis}$,(5.85),(5.86) we note that once $\lambda_h^n, z_h^n, g_h^n$ are known, two Dirichlet problems are necessary to compute $A_h z_h^n$. Once this vector is known we can compute ρ_n, λ_h^{n+1}, g_h^{n+1} by (5.82), (5.83), $(5.84)_{bis}$; then the knowledge of g_h^{n+1} enables us to compute γ_n and z_h^{n+1} by (5.85),(5.86)

Remarque 5.8. : The Remark 5.7 on the prefactorization of the matrices also holds in this case.

Remark 5.9. : Algorithm (5.79)-(5.83),$(5.84)_{bis}$(5.85),(5.86) is more sensitive to the round-off errors than algorithm (5.79)-(5.86) in which g_h^{n+1} is computed by (5.89). Therefore it is reasonnable to use on the former algorithm a periodic reinitialization procedure of the type $z_h^{n+1} = g_h^{n+1}$, g_h^{n+1} being computed by (5.84) instead of $(5.84)_{bis}$.

6. - The case Ω p-connected ($p \geq 1$). (I) The continuous problem.

6.1. Formulation of the problem.

Let $\{\Omega_k\}_{k=0}^{p}$ be a family of simply connected, bounded domains of $\mathbb{R}^2$ with a smooth boundary Γ_k, $k=0,1,\ldots p$. We assume also (see Figure 6.1.) that

$$(6.1) \qquad \bar{\Omega}_k \subset \Omega_o \quad \forall\, k=1,\ldots p.$$

We define then Ω and Γ by

$$(6.2) \qquad \Omega = \Omega_o - \bigcup_{k=1}^{p} \bar{\Omega}_k \ , \ \Gamma = \partial\Omega.$$

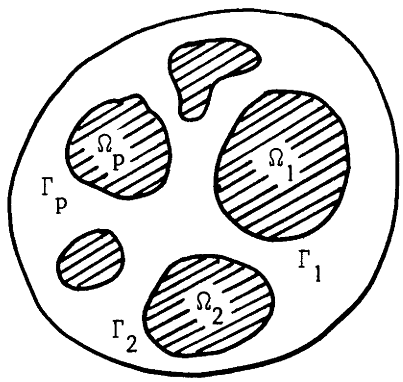

Figure 6.1.

We consider over Ω the following Stokes problem

$$(6.3) \quad \begin{cases} -\mu\Delta\vec{u} + \nabla\vec{p} = \vec{f} \text{ over } \Omega, \\ \nabla\cdot\vec{u} = 0 \text{ over } \Omega, \\ \vec{u}|_{\Gamma_o} = \vec{u}_b \text{ with } \int_{\Gamma_0} \vec{u}_b \cdot \vec{n}\, d\Gamma = 0 \ , \\ \vec{u}|_{\Gamma_k} = 0 \quad \forall\, k=1,\ldots p. \end{cases}$$

Let us introduce

$$(6.4) \qquad V_o = \{\vec{v} \in H^1_o(\Omega) \times H^1_o(\Omega)\ ,\ \nabla\cdot\vec{v} = 0 \text{ p.p. on } \Omega\},$$

$$(6.5) \qquad \begin{cases} V_b = \{\vec{v} \in H^1(\Omega) \times H^1(\Omega)\ ,\ \nabla\cdot\vec{v} = 0 \text{ p.p. on } \Omega, \\ \vec{v}|_{\Gamma_0} = \vec{u}_b\ ,\ \vec{v}\|_{\Gamma_k} = 0 \quad \forall k=1,\dots p\}. \end{cases}$$

In (6.5) we assume that $\vec{u}_b \in H^{1/2}(\Gamma_o) \times H^{1/2}(\Gamma_o)$. If $\vec{f}$ is sufficiently smooth then (6.3) has the following variational formulation

$$(6.6) \qquad \begin{cases} \mu\int_\Omega \nabla\vec{u}\cdot\nabla\vec{v}\ dx = \int_\Omega \vec{f}\cdot\vec{v}dx \qquad \forall \vec{v} \in V_0 \\ \vec{u} \in V_b \end{cases}$$

where $\vec{f}\cdot\vec{v}$ denotes the usual scalar of $\vec{f}$ and $\vec{v}$ in $\mathbb{R}^2$. It follows from, e.g. LIONS [31] that (6.6) has a unique solution.

6.2. <u>A stream function formulation.</u>

From the boundary conditions in (6.3) there exists a stream function ψ such that

$$(6.7) \qquad u_1 = \frac{\partial\psi}{\partial x_2}\ ,\quad u_2 = -\frac{\partial\psi}{\partial x_1} \quad \text{in } \Omega,$$

$$(6.8) \qquad \psi(x) = \int_{\widehat{x_o x}} \vec{u}_b\cdot\vec{n}\ d\Gamma_o \quad \forall x \in \Gamma_o\ ,$$

$$(6.9) \qquad \psi = \text{const.} \quad \text{on} \quad \Gamma_k\ ,\ \forall k=1,\dots p,$$

$$(6.10) \qquad \frac{\partial\psi}{\partial n}\Big|_{\Gamma_0} = -\vec{u}_b\cdot\vec{\tau},$$

$$(6.11) \qquad \frac{\partial\psi}{\partial n}\Big|_{\Gamma_k} = 0 \quad \forall k=1,\dots p.$$

Moreover ψ is the unique solution of the following variational problem

$$(6.12) \qquad \begin{cases} \mu\int_\Omega \Delta\psi\Delta\phi dx = \int_\Omega \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\phi dx + \int_\Gamma (f_1 n_2 - f_2 n_1)\phi d\Gamma \\ \forall\ \phi \in W_o\ ,\ \psi \in W_b \end{cases}$$

where, in (6.12), $n_i = \cos(\vec{n}, 0x_i)$ and

$$(6.13)\quad \begin{cases} W_o = \{\phi \in H^2(\Omega)\ ,\ \frac{\partial\phi}{\partial n}\big|_\Gamma = 0\ ,\ \phi\big|_{\Gamma_o} = 0\ ,\ \phi\big|_{\Gamma_k} = \text{const.} \\ \forall k=1,\dots p\}, \end{cases}$$

$$(6.14)\quad \begin{cases} W_b = \{\phi \in H^2(\Omega)\ ,\ \frac{\partial\phi}{\partial n}\big|_{\Gamma_o} = -\vec{u}_b\cdot\vec{\tau}\ ,\ \frac{\partial\phi}{\partial n}\big|_{\Gamma_k} = 0\ \forall k=1,\dots p\ , \\ \phi\big|_{\Gamma_o} = \int_{\widehat{x_o x}} \vec{u}_b\cdot\vec{n}\,d\Gamma_o\ ,\ \phi\big|_{\Gamma_k} = \text{const.}\ \ \forall k=1,\dots p\}. \end{cases}$$

In (6.9) the constants are unknown. They are arbitrary in (6.13),(6.14). Let us define ϕ_k by $\phi_k = \phi|_{\Gamma_k}$, k=1,...p. It follows from (6.12)-(6.14) thet (6.12) can be reformulated

$$(6.15)\quad \begin{cases} \mu \int_\Omega \Delta\psi\Delta\phi dx = \int_\Omega \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\phi dx + \sum_{k=1}^{p} \phi_k \int_{\Gamma_k} (f_1 n_2 - f_2 n_1)\,d\Gamma_k \\ \forall \phi \in W_o\ ,\psi \in W_b\ . \end{cases}$$

It follows from (6.15) that ψ is also the unique solution of the following <u>minimization problem</u>

$$(6.16)\quad \underset{\phi \in W_b}{\text{Min}}\ J(\phi)$$

where

$$(6.17)\quad J(\phi) = \frac{\mu}{2}\int_\Omega |\Delta\phi|^2 d\,x - \int_\Omega \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\phi dx - \sum_{k=1}^{p} \phi_k \int_{\Gamma_k} (f_1 n_2 - f_2 n_1)\,d\Gamma_k .$$

6.3. <u>A generalized biharmonic problem.</u>

The problem (6.15)) (6.16) is actually a particular case of the slightly more general biharmonic problem

$$(6.18)\quad \begin{cases} \int_\Omega \Delta\hat{\psi}\Delta\phi dx = \int_\Omega f\phi dx + \sum_{k=1}^{p} \gamma_k\phi_k \quad \forall \phi \in W_0\ , \\ \hat{\psi} \in W_g, \end{cases}$$

where

$$(6.19)\quad W_g = \{\phi \in H^2(\Omega)\ ,\ \frac{\partial\phi}{\partial n}\big|_\Gamma = g_2,\ \phi\big|_{\Gamma_o} = g_{10},\ \phi\big|_{\Gamma_k} = g_{1k} + \text{const.}\ \ \forall k=1,\dots p\}.$$

In (6.19) we have $g_{1k} \in H^{3/2}(\Gamma_k)$ $\forall k=0,1,\dots p$, $g_2 \in H^{1/2}(\Gamma)$ and the constants are arbitrary ; (6.18) has a unique solution.
It follows from (6.18),(6.19) that $\hat{\psi}$ is also the unique solution of

(6.20) $$\underset{\psi \in W_g}{\text{Min}} \; J(\phi)$$

where

$$J(\phi) = \frac{1}{2}\int_\Omega |A\phi|^2 dx - \int_\Omega f\phi dx - \sum_{k=1}^{p} \gamma_k C_k ,$$

with $\phi|_{\Gamma_k} = g_{1k} + C_k \; \forall k=1,\dots p.$

6.4. An equivalent formulation of (6.18),(6.20).

In order to reduce (6.18),(6.20) to a set of ordinary biharmonic problems(*) the fundamental result is given by

Theorem 6.1. : Let us define $\hat{C} \in \mathbb{R}^p$ by

(6.21) $$\hat{C}_k = \hat{\psi}|_{\Gamma_k} - g_{1k} , \; k=1,\dots p,$$

where $\hat{\psi}$ is the solution of (6.18),(6.20). Then $\hat{C}$ is the unique solution of

(6.22) $$\begin{cases} j(\hat{C}) \le j(C) & \forall C \in \mathbb{R}^p, \\ \hat{C} \in \mathbb{R}^p, \end{cases}$$

with

(6.23) $$j(C) = \frac{1}{2}\int_\Omega |\Delta\psi|^2 dx - \int_\Omega f\psi dx - \sum_{k=1}^{p} \gamma_k C_k$$

where, in (6.23), ψ is the solution in $H^2(\Omega)$ of the ordinary biharmonic problem

(6.24) $$\begin{cases} \Delta^2\psi = f \text{ over } \Omega, \\ \psi|_{\Gamma_o} = g_{10} , \; \psi|_{\Gamma_k} = g_{1k}+C_k \quad \forall k=1,\dots p, \\ \frac{\partial\psi}{\partial n}|_\Gamma = g_2. \end{cases}$$

(*) i.e. like (P_o) of Sec. 1.

Proof : Let $C \in \mathbb{R}^p$ and let ψ be the corresponding solution of (6.24). Then

$$(6.25) \qquad \psi \in W_g$$

and

$$(6.26) \qquad j(C) = J(\psi) \geq \underset{\phi \in W_g}{\text{Min}}\ J(\phi) = J(\hat{\psi}) \qquad \forall C \in \mathbb{R}^p.$$

Conversely, since $H_o^2(\Omega) \subset W_o$ we obviously have from (6.18)

$$(6.27) \qquad \int_\Omega \Delta\hat{\psi}\Delta\phi dx = \int_\Omega f\phi dx \qquad \forall \phi \in H_o^2(\Omega)$$

which implies

$$(6.28) \qquad \Delta^2\hat{\psi} = f \text{ over } \Omega.$$

Moreover, since $\hat{\psi} \in W_g$ we have

$$(6.29) \qquad \hat{\psi}|_{\Gamma_o} = g_{10}\ , \quad \frac{\partial\hat{\psi}}{\partial n}|_\Gamma = g_2$$

and (6.21) implies

$$(6.30) \qquad \hat{\psi}|_{\Gamma_k} = g_{1k}+\hat{C}_k \quad \forall\ k=1,\dots p.$$

It follows from (6.23),(6.24),(6.28)-(6.30) that

$$(6.31) \qquad J(\hat{\psi}) = j(E) \geq \underset{C \in \mathbb{R}^p}{\inf}\ j(C).$$

Comparing (6.26)) (6.31) we obtain (6.22) and the uniqueness is obvious.∎

Remark 6.1 : The minimization problem (6.22) may be viewed as an optimal control problem in which the control variable is C, the state variable is ψ , the state equation is (6.24) and the cost function is defined by (6.23) .∎

The following result is an obvious consequence of Theorem 6.1. and relations (6.23), (6.24).

Pronosition 6.1 : The minimization nroblem (6.22) has a unique solution which is also the solution of the linear svstem

(6.32) $$\frac{\partial j}{\partial C_k}(\hat{C}) = 0 \ , \quad 1 \leq k \leq p \ ,$$

the matrix of which is symmetric and positive definite.m

6.5. - Mathematical expression of ∇j and application to the solution of (6.18),(6.20).

6.5.1. Expression of ∇j.

In order to solve (6.18),(6.20) through (6.22) the following results are fundamental

Theorem 6.2 : Let ψ be the solution of (6.24) and $\omega = -\Delta\psi$. Then if $j(\cdot)$ is defined by (6.23),(6.24) we have

(6.33) $$\frac{\partial j}{\partial C_k}(C) = \int_{\Gamma_k} \frac{\partial \omega}{\partial n}\, d\Gamma_k - \gamma_k \ , \quad k=1,\dots p.$$

Proof : Let δC ● R', then

(6.34) $$\nabla j\,(C)\cdot\delta C = \lim_{\substack{t\to o \\ t\neq 0}} \frac{j(C+t\delta C)-j\,(C)}{t} = \int_\Omega \Delta\psi\Delta\delta\psi dx - \int_\Omega f\delta\psi dx - \sum_{k=1}^{p} \gamma_k \delta C_k$$

where, in (6.34), $\delta\psi$ is the unique solution in $H^2(\Omega)$ (and W_o) of

(6.35) $$\begin{cases} \Delta^2\delta\psi = 0 \text{ over } \Omega, \\ \delta\psi|_{\Gamma_o} = 0 \ , \ \delta\psi|_{\Gamma_k} = \delta C_k \ \forall\, k=1,\dots p, \\ \frac{\partial}{\partial n}\delta\psi|_\Gamma = 0. \end{cases}$$

It follows from Green's formula that

(6.36) $$\int_\Omega \Delta\psi\Delta\delta\psi dx = \int_\Omega \Delta^2\psi\delta\psi dx + \int_\Gamma \Delta\psi \frac{\partial}{\partial n}\delta\psi dx - \int_\Gamma \frac{\partial}{\partial n}\Delta\psi\delta\psi d\Gamma \ .$$

Since $\Delta^2\psi = f$ and $\omega = -\Delta\psi$ it follows from (6.34)-(6.36) that

$$\nabla j(C)\cdot\delta C = \sum_{k=1}^{p} \int_{\Gamma_k} \left(\frac{\partial\omega}{\partial n}\, d\Gamma_k - \gamma_k\right) \delta C_k \qquad \forall\, \delta C \in \mathbb{R}^p, \tag{6.37}$$

which proves (6.33).■

<u>Remark 6.2</u> : Formula (6.33) is not correct since, usually, $\frac{\partial\omega}{\partial n}\big|_\Gamma$ is not a function but an element of $H^{-3/2}(\Gamma)$. Actually the correct expression for $\frac{\partial j}{\partial C_k}$ is

$$\frac{\partial j}{\partial C_k}(C) = \ll \chi_k, \frac{\partial\omega}{\partial n} \gg - \gamma_k \ , \quad k=1,\dots p \tag{6.38}$$

where χ_k is the function defined over Γ such that

$$\chi_k\big|_{\Gamma_\ell} = \delta_{k\ell} \ , \quad \ell = 0,1,\dots p, \tag{6.39}$$

and where $\ll\cdot,\cdot\gg$ denotes the duality between $H^{3/2}(\Gamma)$ and $H^{-3/2}(\Gamma)$. To prove (6.38) we should use Green's formula (2.2) (see Sec. 2.3) instead of (6.36). ■

<u>Remark 6.3</u> : Let us denote by $\tilde{\chi}_k$ an extension of χ_k over Ω such that $\tilde{\chi}_k$ ● Hi(Q). Then from Green's formula we have

$$\begin{cases} \dfrac{\partial j}{\partial C_k}(C) = \displaystyle\int_\Omega \Delta\omega\tilde{\chi}_k dx + \int_\Omega \nabla\omega\cdot\nabla\tilde{\chi}_k dx - \gamma_k = \\ \qquad = \displaystyle\int_\Omega \nabla\omega\cdot\nabla\tilde{\chi}_k dx - \int_\Omega f\tilde{\chi}_k dx - \gamma_k \ , \quad k=1,\dots p. \end{cases} \tag{6.40}$$

The advantage of (6.40) by comparison to (6.33) is that it gives an expression of $\frac{\partial j}{\partial C_k}$ in which $\frac{\partial\omega}{\partial n}\big|_\Gamma$ <u>does not occur explicitly.</u> This is an important remark in view of the <u>approximate problem</u>.

6.5.2. <u>Application to the solution of</u> (6.18),(6.20).

There are sevaral methods for solving (6.32) ; we can use either <u>direct</u> methods or <u>iterative</u> methods. As in Sec. 5, 6 we can use gradient or conjugate gradient methods, <u>without knowing</u> the matrix of the system (6.32). However, since p is usually <u>small</u> it can be convenient to compute the matrix and the right hand side of (6.32).

We have

(6.41) $$j'(C) = BC-(d+y)$$

where $\gamma = \{\gamma_k\}_{k=1}^{p}$, $d \in \mathbb{R}^p$ and where B is a p×p symmetric positive definite matrix ; B and d are not known a priori. Concerning $\{B,d\}$ we can easily prove the following

Proposition 6.2: We have

(6.42) $$d_k = -\int_{\Gamma_k} \frac{\partial \omega_d}{\partial n} d\Gamma_k \ , \ k=1,\dots p,$$

where $\omega_d = -\Delta\psi_d$, ψ_d being the solution of

(6.43) $$\begin{cases} \Delta^2\psi_d = f \text{ over } \Omega, \\ \psi_d|_{\Gamma_k} = g_{1k} \ \forall k=0,1,\dots p, \\ \frac{\partial \psi_d}{\partial n}|_{\Gamma} = g_2. \ \blacksquare \end{cases}$$

Proposition 6.3: Let $B = \{b_{k\ell}\}_{1\le k,\ell\le p}$, then

(6.44) $$b_{k\ell} = \int_{\Gamma_k} \frac{\partial \omega_\ell}{\partial n} d\Gamma_k \ , \ 1 \le k,\ell \le p,$$

where, $\omega_\ell = -\Delta\psi_\ell$, ψ_ℓ being the solution of

(6.45) $$\begin{cases} \Delta^2\psi_\ell = 0 \text{ over } \Omega, \\ \psi_\ell|_{\Gamma_k} = \delta_{k\ell} \ \forall k=0,1,\dots p, \\ \frac{\partial \psi_\ell}{\partial n}|_{\Gamma} = 0 . \ \blacksquare \end{cases}$$

Remark 6.4 : Remarks 6.2, 6.3 hold for (6.42) and Remark 6.3 holds for (6.44). It follows in particular from (6.40),(6.44),(6.45) that

(6.46) $$b_{k\ell} = \int_\Omega \nabla \omega_\ell \cdot \nabla \tilde{\chi}_k \, dx . \ \blacksquare$$

Remark 6.5 : Since B is symmetric we also have

$$b_{k\ell} = \int_\Omega \nabla\omega_k \cdot \nabla\tilde{\chi}_\ell \, dx \ , \ 1 \le k,\ell \le p.$$

In fact, from the symmetry of B it is convenient to construct B, column by column, by computing only $b_{k\ell}$ for the pairs $\{k,\ell\}$ such that $1 \le \ell \le k \le p$.■

Once B and k are known, solving (6.32), i.e.

$$(6.47) \quad B\hat{C} = d+\gamma$$

is a trivial task which produces $\hat{C}$. Once $\hat{C}$ is known, we obtain $\hat{\psi}$ from (6.24).■

Remark 6.6 : The solution of the "generalized" biharmonic problem (6.18),(6.20), through the solution of (6.32),(6.47) by a direct method, requires the solution over Ω of (p+2) "ordinary" biharmonic problems :

- 1 to compute d,
- p to compute B,
- 1 to compute $\hat{\psi}$ from $\hat{C}$ (this last one is (6.24)).

If we want to solve these ordinary biharmonic problems, using the decomposition (2.26)-(2.30), studied in Sec. 2, we shall have to solve (p+2) "integral equations" like (2.28) and 2(p+3) Dirichlet problems for $-\Delta$ (a superficial analysis would indicate 4(p+2) Dirichlet problems).■

Remark 6.7 : The above matrix B depends of Ω only, therefore it remains unchanged if f, g_{1k} (k=0,1,...p), g_2 are modified. It can be constructed once and for all for a given Ω. ■

6.6. A saddle-point property.

We use the notation of Sec. 6.4 ; taking $\frac{\partial\psi}{\partial n} - g_2$ as a linear constraint, let us define a lagrangian $\mathcal{L} : \mathbb{R}^p \times H^{-1/2}(\Gamma) \to \mathbb{R}$ by

$$(6.48) \quad \mathcal{L}(C,\mu) = \frac{1}{2}\int_\Omega |\Delta\psi|^2 dx - \int_\Omega f\psi dx + \langle\mu, \frac{\partial\psi}{\partial n} - g_2\rangle - \sum_{k=1}^{p}\gamma_k C_k,$$

where in (6.48), $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ and where ψ is a function of C and μ via

$$(6.49)\quad \begin{cases} \Delta^2\psi = f \text{ over } \Omega, \\ \psi|_{\Gamma_o} = g_{10} \;, \psi|_{\Gamma_k} = g_{1k} + C_k \quad \forall k=1,\dots p, \\ -\Delta\psi|_{\Gamma} = \mu. \end{cases}$$

Let us prove

Proposition 6.4. : Let ψ be the solution of (6.49) and $\omega = -\Delta\psi$; we have then

$$(6.50)\quad \frac{\partial \mathcal{L}}{\partial C_k}(C,\mu) = \int_{\Gamma_k} \frac{\partial \omega}{\partial n}\, d\Gamma_k - \gamma_k \quad \forall k=1,\dots p,$$

$$(6.51)\quad \frac{\partial \mathcal{L}}{\partial \mu}(C,\mu) = \frac{\partial \psi}{\partial n} - g_2.$$

Proof : From (6.48), (6.491, we have :

$$(6.52)\quad \begin{aligned} &\frac{\partial \mathcal{L}}{\partial C}(C,\mu)\cdot\delta C + \frac{\partial \mathcal{L}}{\partial \mu}(C,\mu)\cdot\delta\mu = \int_\Omega \Delta\psi\Delta\delta\psi dx - \int_\Omega f\delta\psi dx + \\ &+ <\mu, \frac{\partial}{\partial n}\delta\psi> + <\delta\mu, \frac{\partial\psi}{\partial n} - g_2> - \sum_{k=1}^{p} \gamma_k\, \delta C_k \;. \end{aligned}$$

$$(6.53)\quad \begin{cases} \Delta^2\delta\psi = 0 \;, \\ \delta\psi|_{\Gamma_o} = 0 \;,\; \delta\psi|_{\Gamma_k} = \delta C_k, \\ -\Delta\delta\psi|_{\Gamma} = \delta\mu. \end{cases}$$

Relation (6.49) and Green's formula yield :

$$(6.54)\quad \int_\Omega \Delta\psi\delta\Delta\psi dx - \int_\Omega f\delta\psi dx + \int_\Gamma \frac{\partial}{\partial n}\Delta\psi\delta\psi d\Gamma + <\mu, \frac{\partial}{\partial n}\delta\psi> = 0.$$

Then using (6.53) and (6.54)

$$(6.55)\quad \begin{cases} \int_\Omega \Delta\psi\delta\Delta\psi dx - \int_\Omega f\delta\psi dx + <\mu, \frac{\partial}{\partial n}\delta\psi> = \int_\Gamma \frac{\partial\omega}{\partial n}\delta\psi\, d\Gamma = \\ \sum_{k=1}^{p} \delta C_k \int_{\Gamma_k} \frac{\partial \omega}{\partial n}\, d\Gamma_k. \end{cases}$$

Therefore :

$$(6.56)\quad \frac{\partial \mathcal{L}}{\partial C}(C,\mu)\cdot \delta C + \frac{\partial \mathcal{L}}{\partial \mu}(C,\mu)\cdot\delta\mu = \sum_{k=1}^{p}\left(\delta C_k \int_{\Gamma_k} \frac{\partial \omega}{\partial n}\, d\Gamma_k - \gamma_k\right) + \langle \delta\mu, \frac{\partial \psi}{\partial n} - g_2 \rangle ,$$

which completes the proof.

Remark 6.8 : Remark 6.2. still holds for the proof of Proposition 6.4 and for the above formulae.

Proposition 6.5 :
Let $\hat{\psi}$ and $\hat{C}$ be respectively the solutions of (6.18),(6.19) and (6.20). Let $\hat{\lambda}$ be equal to $\hat{\omega}|_\Gamma$, then $\{\hat{C},\hat{\lambda}\}$ is the unique saddle-point of $\mathcal{L}$ over $\mathbb{R}^p \times H^{-1/2}(\Gamma)$.

Proof : From (6.29), (6.32), (6.33), (6.50), (6.51) we have

$$(6.57)\quad \frac{\partial \mathcal{L}}{\partial C}(\hat{C},\hat{\lambda}) = 0 ,$$

$$(6.58)\quad \frac{\partial \mathcal{L}}{\partial \mu}(\hat{C},\hat{\lambda}) = 0.$$

Then to prove that $\{\hat{C},\hat{\lambda}\}$ is a saddle point of $\mathcal{L}$ over $\mathbb{R}^p \times H^{-1/2}(\Gamma)$, it is sufficient to show that $\mathcal{L}$ is convex in C and concave in μ ; and a necessary and sufficient condition for this is

$$(6.59)\quad \left(\frac{\partial \mathcal{L}}{\partial C}(C+\delta C,\mu) - \frac{\partial \mathcal{L}}{\partial C}(C,\mu)\right)\cdot \delta C \geq 0 \quad \forall \delta C \in \mathbb{R}^p ,\forall \mu \in H^{-1/2}(\Gamma) ,$$

$$(6.60)\quad \langle \frac{\partial \mathcal{L}}{\partial \mu}(C,\mu+\delta\mu) - \frac{\partial \mathcal{L}}{\partial \mu}(C,\mu),\delta\mu \rangle \geq 0 \ \forall \delta\mu \in H^{-\frac{1}{2}}(\Gamma),\ \forall C \in \mathbb{R}^p.$$

From (6.50) we must show that :

$$(6.61)\quad \sum_{k=1}^{p} \delta C_k \int_{\Gamma_k} \frac{\partial \delta\omega}{\partial n}\, d\Gamma_k \geq 0 \qquad \forall \delta C \in \mathbb{R}^p$$

where $\delta\omega = -\Delta\delta\psi$, and

$$(6.62)\quad \begin{cases} \Delta^2 \delta\psi = 0 , \\ \delta\psi|_{\Gamma_o} = 0 ,\ \delta\psi|_{\Gamma_k} = \delta C_k ,\ k=1,\dots,p, \\ -\Delta\delta\psi|_\Gamma = 0. \end{cases}$$

Green's formula and (6.62) imply that

$$0 = \int_\Omega \Delta^2 \delta\psi \; \delta\psi dx = \int_\Omega |\Delta\delta\psi|^2 dx + \int_\Gamma \frac{\partial}{\partial n} \Delta\delta\psi\delta\psi d\Gamma \; .$$

Therefore

$$\int_\Gamma \frac{\partial\delta\omega}{\partial n} \delta\psi d\Gamma = \sum_{k=1}^{p} \delta C_k \int_{\Gamma_k} \frac{\partial\delta\omega}{\partial n} d\Gamma_k = \int_\Omega |\Delta\delta\psi|^2 dx \geq 0 \; .$$

The proof of (6.60) is almost similar ; we leave it to the reader. ∎

From Proposition 6.5 and the convex-concave property of $\mathcal{L}$ it follows from GLOWINSKI-LIONS-TREMOLIERES [23, Ch. 21 , FORTIN-GLOWINSKI [20] that for solving (6.18)-(6.20) we can use the following algorithm of Arrow-Hurwicz type (*) :

(6.63) $\{C^o, \lambda^o\} \in \mathbb{R}^p \times H^{-1/2}(\Gamma)$, arbitrarily given,

then for $n \geq 0$

(6.64) $\lambda^{n+1} = \lambda^n + \rho_1 S^{-1} \frac{\partial\mathcal{L}}{\partial\mu}(C^n, \lambda^n)$, $\rho_1 > 0$,

(6.65) $C^{n+1} = C^n - \rho_2 \frac{\partial\mathcal{L}}{\partial C}(C^n, \lambda^{n+1})$, $\rho_2 > 0$.

In (6.64), S is a duality mapping from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$. It is convenient to write (6.64), (6.65) in the following equivalent form which is more suitable for computations :

(6.66) $\begin{cases} -\Delta\omega^{n+1} = f \text{ over } \Omega \\ \omega^{n+1}|_\Gamma = \lambda^n, \end{cases}$

(6.67) $C_k^{n+1} = C_k^n - \rho_2 \left(\int_{\Gamma_k} \frac{\partial\omega^{n+1}}{\partial n} d\Gamma_k - \gamma_k \right)$,

(6.68) $\begin{cases} -\Delta\psi^{n+1} = \omega^{n+1} \\ \psi^{n+1}|_{\Gamma_o} = g_{10} \; , \; \psi^{n+1}|_{\Gamma_k} = g_{1k} + C_k^{n+1} \quad \forall k=1,\dots p, \end{cases}$

(*) we only consider an algorithm with constant step ρ_1, ρ_2.

(6.69) $\lambda^{n+1} = \lambda^n + \rho_1 S^{-1}\left(\frac{\partial\psi^{n+1}}{\partial n} - g_2\right).$

Remark 6.9 : If $\hat{\lambda} \in L^2(\Gamma)$ then S can be replaced by I in (6.69). ■

Remark 6.10: The above algorithm is a precise formalization of some of the concepts felt by PERRONNET C381.

Remark 6.11 : Thus the biharmonic problem on a multiconnected domain has been replaced by a sequence of Dirichlet problems for Laplace's operator.

7. - THE CASE Ω p-CONNECTED ($p \geq 1$). (II) THE DISCRETE CASE.

7.1. Formulation of the approximate problem.

We assume in this section that Ω_k is a polygonal $\forall k=0,1,\ldots p$. The spaces V_h, V_{oh}, $\mathcal{M}_h$ being defined as in Sec. 3.1, we define W_{gh} by

$$\begin{cases} W_{gh} = \{(v_h,q_h) \in V_h \times V_h,\ v_h|_{\Gamma_0} = g_{1oh},\ v_h|_{\Gamma_k} = g_{1kh} + \text{const.}\ \forall k=1,\ldots p, \\ \int_\Omega \nabla v_h \cdot \nabla \mu_h dx = \int_\Omega q_h \mu_h dx + \int_\Gamma g_{2h}\, \mu_h\, d\Gamma\ \forall \mu_h \in V_h\}. \end{cases}$$

We approximate (6.18),(6.20) by

(7.1) $\displaystyle\min_{(v_h,q_h) \in W_{gh}} J_h(v_h,q_h)$

where

(7.2) $J_h(v_h,q_h) = \frac{1}{2}\int_\Omega |q_h|^2 dx - \int_\Omega f_h v_h dx - \sum_{h=1}^{p} \gamma_k C_k$

with $v_h|_{\Gamma_k} = g_{1kh} + C_k\ \forall k=1,\ldots p$.

The approximate problem (7.1) has a unique solution $\{\hat{\psi}_h,\hat{\omega}_h\}$ and it is also equivalent to

(7.3) $\displaystyle\min_{C \in \mathbb{R}^p} j_h(C)$

with

$$(7.4)\quad j_h(C) = \frac{1}{2}\int_{\Omega} |\omega_h|^2 dx - \int_\Omega f_h \psi_h dx - \sum_{k=1}^{p} \gamma_k C_k$$

where in (7.4) $\{\psi_h, \omega_h\}$ depends on C via the "state problem"

$$(7.5)\quad \min_{(v_h,\, q_h) \in W_{gh}(C)} J_{oh}(v_h, q_h)$$

in which

$$(7.6)\quad W_{gh}(C) = \{(v_h, q_h) \in W_{gh}\ ,\ v_h|_{\Gamma_k} = g_{1kh} + C_k\ \forall k=1,\ldots p\}$$

and

$$(7.7)\quad J_{oh}(v_h, q_h) = \frac{1}{2}\int_\Omega |q_h|^2 dx - \int_\Omega f_h v_h dx\ .$$

Clearly we have the following

<u>Proposition 7.1</u> : <u>The minimisation problem</u> (7.3) <u>has a unique solution</u> $\hat{C}_h$ <u>which is also the solution of the linear system</u>

$$(7.8)\quad \frac{\partial j_h}{\partial C_k}(\hat{C}_h) = 0\ ,\ 1 \le k \le p$$

<u>the matrix of which is symmetric and positive definite.</u>

7.2. <u>Solution of (7.1) via (7.3)</u>.

7.2.1. <u>Computation of</u> ∇j_h.

We begin by stating the following

<u>Proposition 7.2</u> : <u>Let</u> $\{\psi_h, \omega_h\}$ <u>be the solution of</u> (7.5)-(7.7), <u>then if</u> $j_h(\cdot)$ <u>is defined by</u> (7.4)-(7.7) <u>we have</u>

$$(7.9)\quad \frac{\partial j_h}{\partial C_k}(C) = \int_\Omega \nabla\omega_h \cdot \nabla\tilde{\chi}_k dx - \int_\Omega f_h \tilde{\chi}_k dx - \gamma_k$$

<u>where</u>

$$(7.10)\quad \begin{cases} \tilde{\chi}_k \in \mathcal{M}_h, \\ \tilde{\chi}_k|_{\Gamma_\ell} = \delta_{k\ell}\ \ \forall \ell=0,1,\ldots p. \end{cases}$$

Proof : From (7.4) we have :

(7.11) $$j'_h(C) \cdot \delta C = \int_\Omega \omega_h \delta\omega_h dx - \int_\Omega f_h \delta\psi_h dx - \sum_{k=1}^{p} \gamma_k \delta C_k.$$

Let us decompose $\delta\psi_h$ into

(7.12) $$\delta\psi_h = \delta\overline{\psi}_h + \sum_{k=1}^{p} \delta C_k \tilde{\chi}_k$$

where

$$\delta\overline{\psi}_h = \delta\psi_h - \sum_{k=1}^{p} \delta C_k \tilde{\chi}_k \text{ belongs to } V_{oh}$$

Then from (3.12) with $v_h = \delta\overline{\psi}_h$, we have :

(7.13) $$\int_\Omega \nabla\omega_h \cdot \nabla\delta\psi_h dx = \int_\Omega f_h \delta\psi_h dx + \sum_{k=1}^{p} \left[\int_\Omega (\nabla\omega_h \nabla\tilde{\chi}_h - f_h \tilde{\chi}_h) dx \right] \delta C_k.$$

Since $\{\psi_h, \omega_h\} \in W_{gh}$, we have

(7.14) $$\int_\Omega \nabla\delta\psi_h \cdot \nabla\omega_h dx = \int_\Omega \delta\omega_h \omega_h dx.$$

Using (7.13)) (7.14) in (7.11) we find the discrete analogue of (6.38)

(7.15) $$j'_h(C) \cdot \delta C = \sum_{k=1}^{p} \delta C_k \left[\int_\Omega \nabla\omega_h \cdot \nabla\tilde{\chi}_k dx - \int_\Omega f_h \tilde{\chi}_k dx - \gamma_k \right]. \blacksquare$$

7.2.2. Application to the solution of (7.1)-(7.3).

As in the continuous case we can solve (7.8) by direct or iterative methods.

7.2.2.1. : Direct method.

We have

(7.16) $$j'_h(C) = B_h C - (d_h + \gamma)$$

where $\gamma = \{\gamma_k\}_{k=1}^{p}$, $d_h \in \mathbb{R}^p$ and where B_h is a $p \times p$ symmetric, positive definite matrix ; B_h and d_h are not known but can be computed from the following

<u>Proposition 7.3</u> : <u>We have</u>

$$(7.17)\qquad d_{hk} = \int_\Omega \nabla\omega_{dh}\cdot\nabla\tilde{\chi}_k + \int_\Omega f_h\,\tilde{\chi}_k dx \;,\quad k=1,\dots p$$

<u>where</u> $\{\psi_h,\omega_{dh}\}$ <u>are the solutions of</u> (7.5),(7.6) <u>with</u> $C=0$.

<u>Proposition 7.4</u> : Let $B_h = \{b_{k\ell}\}_{1\le k,\ell\le p}$, then

$$(7.18)\qquad b_{k\ell} = \int_\Omega \nabla\omega_{h\ell}\cdot\nabla\tilde{\chi}_k dx$$

<u>where</u> $\{\psi_{h\ell},\omega_{h\ell}\}$ <u>is the solution of</u> (7.5),(7.6) <u>with</u>

$$f_h = 0,\; g_{1kh} = 0\;\forall k=0,\dots p,\; g_{2h} = 0\;,\; C_k = \delta_{k\ell}\;.$$

<u>Remark 7.1</u> : Remarks 6.5, 6.6, 6.7 hold.

7.2.2.2 : <u>Iterative methods</u>.

As in Sec. 5,6 we can use gradient or conjugate gradient methods to solve (7.8) without computing explicitly B_h and d_h.

Moreover as in the continuous case, an alternative method would be to compute the saddle-point in $\mathbb{R}^p\times\mathcal{M}_h$ of

$$(7.19)\quad \begin{cases} \mathcal{L}_h(C,\mu_h) = \frac{1}{2}\int_\Omega |\omega_h|^2 dx - \int_\Omega f_h\psi_h dx + \int_\Omega \nabla\mu_h\cdot\nabla\psi_h dx - \int_\Omega \mu_h\omega_h dx - \\ \qquad -\int_\Gamma \mu_h\, g_{2h}\, d\Gamma - \sum_{k=1}^{p}\gamma_k\, C_k, \end{cases}$$

where $\{\omega_h,\psi_h\}$ is a function of C and μ_h via

$$\begin{cases} \int_\Omega \nabla\omega_h\cdot\nabla v_h dx = \int_\Omega f_h v_h dx \quad \forall\, v_h\in V_{oh} \\ \omega_h-\mu_h\in V_{oh}, \end{cases}$$

$$\begin{cases} \int_\Omega \nabla\psi_h\cdot\nabla v_h dx = \int_\Omega \omega_h v_h dx \quad \forall\, v_h\in V_{oh} \\ \psi_h\in V_h,\; \psi_h|_{\Gamma_o} = g_{oh}\;,\; \psi_h|_{\Gamma_k} = g_{1kh} + C_k. \end{cases}$$

The reader will have no difficulties in finding the discrete analogues of (6.50), (6.51) and of the Arrow-Hurwicz algorithm (6.64)-(6.69).

Remark 7.2 : If $\mathcal{T}_h$ is chosen as in (4.2), cf. Sec. 4.2, the above integrals in (7.17)-(7.19) are in fact to be done on the boundary triangles only. ∎

8. Further Remarks. Comments.

Remark 8.1 : Various $s_h(\cdot,\cdot)$ have been given in Sec. 5.2.3., 5.2.5. The corresponding matrices Sh are symmetric and positive definite. In view of iterating in $H^{-1/2}(\Gamma)$, "approximately", we feel that a good strategy is to choose S_h as the inverse matrix of the matrix related to (5.19). Numerical experiments to test this conjecture are planned for the near future. ∎

Remark 8.2 : In the conjugate gradient method of Sec. 5.4 we have used the canonical inner-product of $\mathbb{R}^{N_h}$. However it is also possible to use an inner-product related to a matrix S_h symmetric and positive definite. The various formulae will be a little more complicated, but the various remarks done in the case of gradient methods about the choice of S_h and s_h still hold for these variants of algorithm (5.79)-(5.86).∎

A large part of the results of this report were announced in GLOWINSKI-PIRONNEAU [25], [26], [27]. In fact this document has to be followed by other reports of GLOWINSKI-PIRONNEAU, BOURGAT-GLOWINSKI-PIRONNEAU, etc.., in which the above results and methods will be extended to the numerical treatment of

- $-\frac{\partial}{\partial t}\Delta\psi + \nu\Delta^2\psi = f,$

with appropriate boundary conditions,

- Navier-Stokes equations,

etc... .

About the choice between the various methods described above, it appears from our numerical experiments that the two most efficient methods are :

(i) The conjugate gradient method of Sec. 5.1 if the approximate biharmonic problem has to be solved only a small number of times and/or if N_h is very large.

(ii) The "quasi-direct" method of Sec. 4 if we need a biharmonic solver to be used a large number of times. It is in particular the case when solving by some iterative methods the Navier-Stokes equations in the $\{\psi,\omega\}$ formulation.

To conclude we would like to point out that a fundamental tool for obtaining these methods is the mixed finite element method of Sec. 3, because its very fascinating (!) algebraic properties.

Some applications of the gradient method with constant step of Sec. 5 may be found in BOURGAT [5].

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