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COMPUTATIONS RELATED TO G-STABILITY OF LINEAR MULTISTEP METHODS

by

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Computations Related to G-Stability of Linear Multistep Methods

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Abstract.-In Dahlquist's recent proof of the equivalence of A-stability and G-stability[1], an algorithm was presented for calculating a G-stability matrix for any A-stable linear multistep method, Such matrices, and various quantities computable from them, are useful in many aspects of the study of the stability of a given method, For example, information may be gained as to the shape of the stability region, or the rate of growth of unstable solutions, We present a summary . of the relevant theory and the results of some numerical calculations performed for several backward differentiation, Adams-Bashforth, and Adams-Moulton methods of low order.

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1. Introduction,

The theory of G-stability arises from investigating the stability of the linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(y_{n+j})$$

for solving the general non-linear system of differential equations

$$y' = f(Y)$$

where $f: \mathbb{C}^s \to \mathbb{C}^s$ satisfies some monotonicity condition

$$\mathrm{Re}\,\langle f(u) - f(v), u - v\rangle \leq 0 \quad \forall u, v \in \quad \mathbb{C}^s. \tag{1.1}$$

Here $\langle \cdot, \cdot \rangle$ is some appropriate inner product in $\mathbb{C}^s$. This condition ensures that the true solution of the differential equation is stable, For let u and v be two solutions of $y' = f(y)$. Then it is easy to see that

$$\frac{d}{dt}\|u(t) - v(t)\|^2 = 2\mathrm{Re}\ (f(u) - f(v), u - v\rangle$$

where $\|\cdot\|$ is the norm corresponding to the given inner product. With the exception of section 5, in this paper we will restrict ourselves to the case $s = 1$ for simplicity. By the theory presented in [1], this means no loss of generality. The inner product is usually simply $(u, v) = \bar{u}v$.

In practice it is often easier to study the stability of the "one-leg method"

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = hf\left(\sum_{j=0}^{k} \beta_j y_{n+j}\right) \tag{1.2}$$

since this involves the function f evaluated at only one point, It has been shown that results for this problem can be easily transformed into results for the corresponding linear multistep method.

For the method (1.2), define the generating polynomials ρ and σ by

$$\rho(\varsigma) = \sum_{j=0}^{k} \alpha_j \varsigma^j,$$

$$\sigma(\varsigma) = \sum_{j=0}^{k} \beta_j \varsigma^j.$$

Wc will often refer to (1.2) as "the method (ρ, σ)". If we define the forward shift operator E by

$$Ey_n = y_{n+1},$$

then we can rewrite (1.2) as

$$\rho(E)y_n = hf(\sigma(E)y_n).$$

Frequently we will use capital letters to denote k-vectors with the convention that

$$Y_n = (y_n, y_{n+1}, \ldots, y_{n+k-1})^T.$$

If G is any real symmetric positive definite matrix, we can define the *G-norm* of the vector Y_n by

$$\|Y_n\|_G^2 = Y_n^H G Y_n.$$

The method (ρ, σ) is termed G-stable if there is a real symmetric positive definite matrix G for which

$$\|Z_1\|_G^2 - \|Z_0\|_G^2 \leq \text{Re}\,\langle \sigma(E)z_0, \rho(E)z_0 \rangle \tag{1.3}$$

for all vectors $Z_n = (z_n, z_{n+1}, \ldots, z_{n+k-1})$, $z_j \in \mathbb{C}$. This will imply stability of the numerical procedure in the following sense. Let $\{y_n'\}$ and $\{y_n''\}$ be two sequences which satisfy (1.2) with different initial conditions, where f is assumed to satisfy (1.1) and (ρ, σ) is G-stable, Then if $z_n = y_n' - y_n''$, it can easily be shown that $\|Z_{n+1}\|_G \leq \|Z_n\|_G$.

It was shown in [1] that G-stability is equivalent to A-stability. So a matrix G satisfying (1.3) exists for a method (ρ, σ) if and only if the A-stability condition holds, Re $\rho(\varsigma)/\sigma(\varsigma) > 0$ for $|\varsigma| > 1$. This is equivalent to requiring that the stability region of the method include the entire left half plane, where the stability region S of a method is defined as the set of complex points q for which the roots of the polynomial $\rho(\varsigma) - q\sigma(\varsigma)$ are inside the unit circle, or lie on the unit circle and are simple roots,

A method for constructing G-stability matrices was originally proposed in [2]. However, that method is not guaranteed to produce positive definite matrices. Nonetheless, it has been successfully used by Dan Andrée and has never failed to produce positive definite matrices in practice. A new algorithm is developed in [1] which is guaranteed to produce positive definite matrices. That algorithm, which will hereafter be referred to as I', has been used to obtain all of the results presented here.

For any A-stable method (ρ,σ), the algorithm I' will gcncratc a complex matrix M such that the real part of $M^H M$ is the required G-matrix, More interesting than the matrix itself, however, are some of the quantities which can be computed from G. These are described in the following sections in which we summarize some of the important results of [1]. The interested reader should refer to that paper for a more detailed discussion of the theory,

Section 7 then contains a summary of some numerical results for the backward differentiation, Adams-Bashforth, and Adams-Moulton methods of various orders.

2. Condition Numbers

One quantity which is of interest to compute is the condition number of the matrix M produced by the algorithm. This is defined by

$$\kappa(M) = ||M||_2 ||M^{-1}||_2.$$

This is important because the theory of G-stability guarantees bounded solutions only in terms of the G-norm, We see that

$$\begin{aligned} ||Y||_G^2 &= Y^H G Y \\ &= Y^H M^H M Y \\ &= ||MY||_2^2. \end{aligned}$$

Hence we know $||MY_{n+1}||_2 \leq ||MY_n||_2 \leq \cdots \leq ||MY_0||_2$. For a bound on the 2-norm of the solution Y_n itself, we have

$$\begin{aligned} ||Y_{n+1}||_2 &= ||M^{-1} M Y_{n+1}||_2 \\ &\leq ||M^{-1}||_2 ||MY_{n+1}||_2 \\ &\leq ||M^{-1}||_2 ||MY_0||_2 \\ &\leq \kappa(M) ||Y_0||_2. \end{aligned}$$

The G-stability of the method might seem somewhat meaningless if it turned out that the algorithm produced matrices M with extremely large condition numbers. In most cases of practical interest the condition number is of moderate size, although in some cases it is on the order of 1000, see section 7.

3. Generalizations of G-stability and the computation *of* b(0).

If a method is G-stable, its stability region contains the entire left half plane, {q: Re $q < 0$}. Not all practical methods are G-stable, however, and we often wish to investigate methods which are not. For such methods there are two questions we may want to consider, Firstly, what sort of **contractivity** condition for f must we replace (1.1) by in order to ensure that (ρ,σ) provides a stable solution when applied to $y' = f(y)$. Secondly, we may wish to know how fast the numerical solution might grow if f fails to satisfy such a condition.

Consider a method (ρ,σ) whose stability region contains the arbitrary "disk" $\{q\colon \mathrm{Re}\left(\frac{aq+b}{cq+d}\right) \leq 0\}$. Define the modified method (ρ^*,σ^*) by

$$
\begin{aligned}
\rho^* &= a\rho + b\sigma \\
\sigma^* &= c\rho + d\sigma,
\end{aligned}
\tag{3.1}
$$

Then the difference equation $\rho(E)y_n = q\sigma(E)y_n$ is equivalent to $\rho^*(E)y_n = q^*\sigma^*(E)y_n$ where $q^* = (aq+b)/(cq+d)$. So clearly (ρ^*, σ^*) is G-stable, since its stability region contains the left half plane, $\{q^*\colon \mathrm{Re}\, q^* \leq 0\}$. In other words,

$$
\mathrm{Re}\,\frac{\rho^*(\varsigma)}{\sigma^*(\varsigma)} > 0 \qquad \text{for} \quad |\varsigma| > 1. \tag{3.2}
$$

In [1] it is shown that applying the method (ρ,σ) to the differential equation $y' = f(y)$ is equivalent to applying (ρ^*,σ^*) to the problem $y' = f^*(y)$, where the modified function f^* is defined by

$$
hf^*(y) = ahf(u(y)) + bu(y)
$$

with u(y) given by the solution of

$$
chf(u) + du = y.
$$

So we will obtain a stable numerical solution provided f^* satisfies the condition (1.1). Hence f must satisfy the condition

$$
\langle ah(f(u) - f(v)) + b(u-v),\ ch(f(u) - f(v)) + d(u-v)\rangle \leq 0 \qquad \forall u, v. \tag{3.3}
$$

In summary, then, we see that (ρ,σ) will provide a stable solution in some norm for the problem $y' = f(y)$ provided that f satisfies (3.3). The matrix defining the norm in question can be obtained by applying the algorithm I' to the modified polynomials ρ^* and σ^* as defined in (3.1).

As a special case we could let $(a, b, c, d) = (1, -m, 0, 1)$ if the stability region were to contain the half plane $\{q\colon \mathrm{Re}\, q \leq m\}$. A method satisfying this condition has been called **(G,m)-stable** in [2]. However, rather than handling (G,m)-stability in this manner, it is preferable to treat it as a special case of a **morc gcncral** situation in which (3.2), which guaranteed the G-stability of (ρ^*, σ^*), is replaced by

$$\text{Re}\ \frac{\rho^*(\varsigma)}{\sigma^*(\varsigma)} > m(\theta) \qquad \text{for } |\varsigma| > \theta. \tag{3.4}$$

(G,m)-stability is then equivalent to the condition that (3.4) be satisfied for (a, b, c, $d) =$ (1, 0, 0, 1) and $m(1) = m$. The reason for considering this generalization is that it is useful in studying the growth of solutions to $y' = f(y)$ when f does not satisfy (3.3). Or, equivalently, when f does not satisfy (1.1).

Define the polynomials ρ^{**} and σ^{**} by

$$\begin{aligned} \rho^{**}(\varsigma) &= \rho^*(\theta\varsigma) - m(\theta)\sigma^*(\theta\varsigma) \\ \sigma^{**}(\varsigma) &= \sigma^*(\theta\varsigma). \end{aligned} \tag{3.5}$$

The algebraic condition (3.4) is then equivalent to

$$\text{Re}\ \frac{\rho^{**}(\varsigma)}{\sigma^{**}(\varsigma)} > 0 \qquad \text{for } |\varsigma| > 1,$$

so ($*,a**) is G-stable, The algorithm I' can be applied to (ρ^{**}, σ^{**}) to yield a positive definite matrix G**(O),

Suppose *now* that f^* satisfies not (1.1) but rather a condition of the form

$$\text{Re}\ \langle f^*(u) - f^*(v), u - v\rangle \le \mu|u - v|^2.$$

It has been shown in [1] that if we then apply a method (ρ^*, σ^*) which satisfies (3.4) to an arbitrary vector Z_n for the equation $y' = f(y)$, we will obtain a new vector Z_{n+1} satisfying

$$\|Z_{n+1}\|^2_{G^*(\theta)} - \theta^2\|Z_n\|^2_{G^*(\theta)} \le 2(\mu h - m(\theta))|\sigma^*(E)z_n|^2. \tag{3.6}$$

where $G^*(\theta) = \Theta^{-1}G^{**}(\theta)\Theta^{-1}$ with $\Theta = \text{diag}(\theta, \theta^2, \ldots, \theta^k)$. Furthermore,

$$2|\sigma^*(E)z_n|^2 \le b(\theta)(\|Z_{n+1}\|^2_{G^*(\theta)} + \theta^2\|Z_n\|^2_{G^*(\theta)}),$$

where

$$b(\theta) = \max_{z_0,\ldots,z_k}\left[\frac{2|\sigma^{**}(E)z_0|^2}{\|Z_1\|^2_{G^{**}(\theta)} + \|Z_0\|^2_{G^{**}(\theta)}}\right].$$

Using this bound in (3.6) when $\mu h > m(0)$ and the bound $|\sigma^*(E)z_n|^2 \geq 0$ when $\mu h \leq m(\theta)$ gives a bound in the $G^*(\theta)$-norm for the growth in the solution of the one leg method:

$$\|Y_{n+1}\|_{G^*(\theta)} \leq \theta' \|Y_n\|_{G^*(\theta)},$$

where

$$\theta' = \begin{cases} \theta, & \text{if } \mu h \leq m(\theta) \\ \theta\left(\frac{1+b(\theta)(\mu h - m(\theta))}{1-b(\theta)(\mu h - m(\theta))}\right)^{1/2}, & \text{if } m(\theta) \leq \mu h \leq m(\theta) + 1/b(\theta). \end{cases} \tag{3.7}$$

The quantity $b(\theta)$ can be calculated in practice as the largest eigenvalue (in modulus) of the generalized eigenvalue problem

$$2(\beta^{**})^H \beta^{**} z = \lambda \left[\begin{pmatrix} 0 & 0 \\ 0 & G^{**}(\theta) \end{pmatrix} + \begin{pmatrix} G^{**}(\theta) & 0 \\ 0 & 0 \end{pmatrix} \right] z,$$

where

$$\beta^{**} = (\beta_0^{**}, \beta_1^{**}, \ldots, \beta_k^{**}),$$

the vector consiting of the coefficients of σ^{**}. Values of $b(\theta)$ are tabulated in section 7.

4. Computation of $m(\theta)$.

For expository purposes we define the region S_θ for a method (ρ, σ) to be the set of complex numbers q such that the polynomial $\rho(\varsigma) - q\sigma(\varsigma)$ has roots of modulus no greater than θ, and only simple roots of modulus θ.

We are often confronted with the problem of trying to determine some of the important characteristics of the region S_θ for some method (ρ, σ). For example, we may want to determine the largest value of m for which the half plane $\{q: \text{Re } q < m\}$ is contained in S_θ. Or we may want to know the diameter of the largest disk contained in both S_θ and the left half plane which is tangent to the imaginary axis at the origin. We refer to these as the *half-plane case* and the disk case respectively.

These and other such questions can be answered by studying the generalized method (ρ^*, σ^*) for a judicious choice of the parameters a, b, c and d in (3.1). If it is assumed that (ρ^*, σ^*) will be $(G, m(\theta))$-stable for some value of $m(\theta)$, we can compute $m(\theta)$ as

$$m(\theta) = \min_{|\varsigma|=\theta} \text{Re } \frac{\rho^*(\varsigma)}{\sigma^*(\varsigma)}.$$

The condition Re $\frac{\rho^*(\varsigma)}{\sigma^*(\varsigma)} > m(\theta)$ for $|\varsigma| > \theta$ will then be satisfied by the minimum principle for harmonic functions. So m(0) can be easily computed by a good one-dimensional minimization routine.

As an example, the two questions posed above are answered for the stability region of the 5-step Backward Differentiation method. Let $\theta = 1$ and let m_1 be the value of m(1) for the half-plane Case, (a, b, c, d) = (1, 0, 0, 1). This turns out to be $m_1 = -2.327$, This is the answer to the first question, since S must contain the half plane {q: Re $q \leq m_1$}. Similarly, in the disk case (a, b, c, d) = (0, 1, 1, 0) and we find that $m_2 = -0.368$. Since S must then contain the disk {q: Re $1/q \leq m_2$}, the diameter we seek is $-1/m_2 = 2.717$.

Figure 1 shows the complement of the region S as well as the two regions determined above. Values of $m(\theta)$ computed for different methods and values of θ are summarized in section 7.

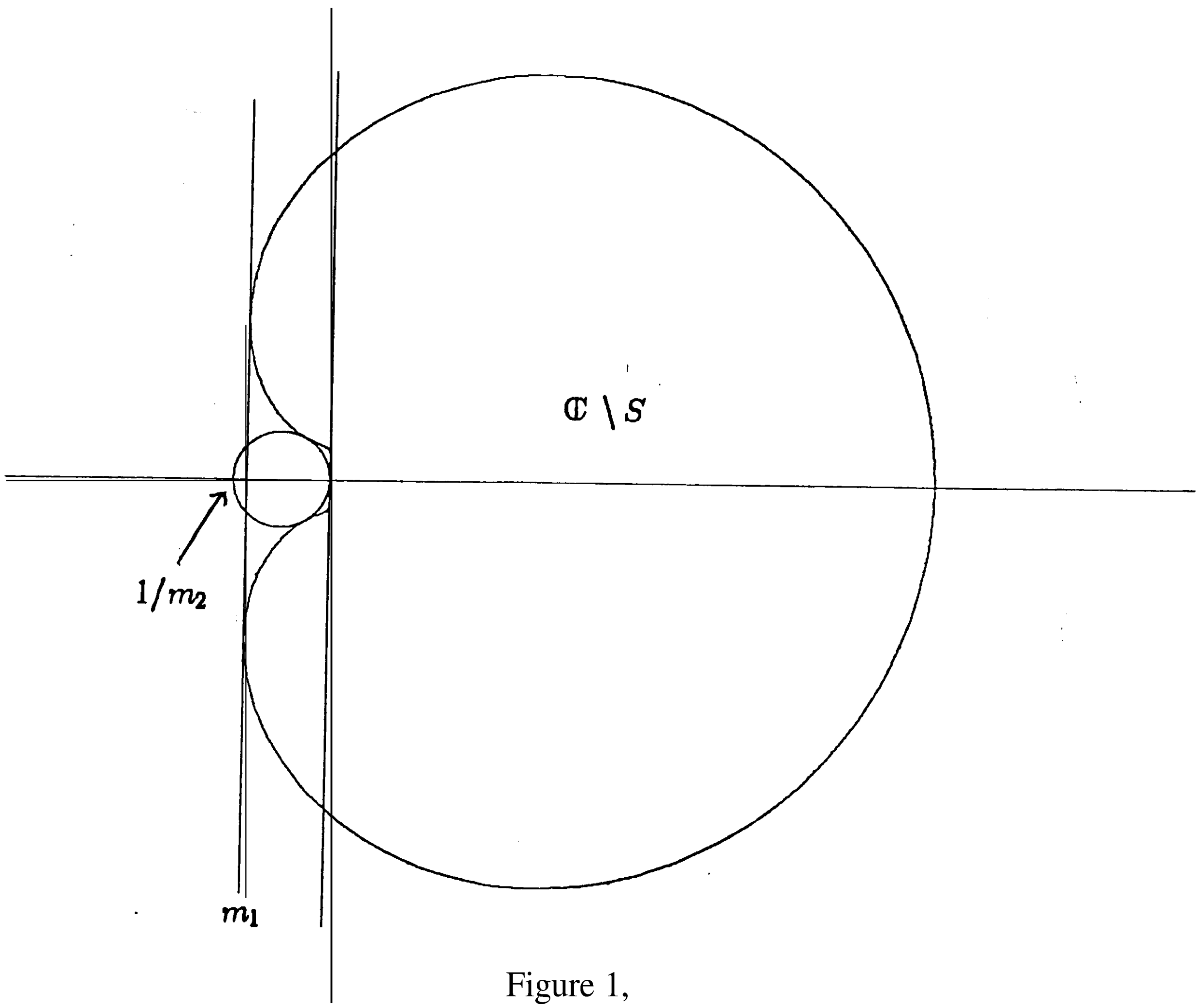


Figure 1,

5. Applications to linear systems with variable coefficients: $c(\theta)$.

Consider now the application of the one-leg method (ρ,σ) to a linear system with variable coefficients,

$$\frac{dy}{dt} = J(t,y)y + p(t).$$

The difference equation becomes

$$\rho(E)y_n = hJ_n\sigma(E)y_n + p_n,$$

where $J_n = J(\sigma(E)t_n, \sigma(E)y_n)$ and $p_n = p(\sigma(E)t_n)$. Here $y \in \mathbb{R}^s$, we are considering s-dimensional systems. In this case, the G-norm of a vector Y_n (each of whose components is now an s-vector) is defined by

$$\|Y_n\|_G^2 = \sum_{i=1}^{k}\sum_{j=1}^{k} g_{i,j}\langle y_{n+i-1}, y_{n+j-1}\rangle.$$

From a result in [1], we know that if $\{y_n\}$ satisfies this one-leg difference equation, then $\{\sigma(E)y_n\}$ will satisfy the corresponding linear multistep difference equation. So we would like to bound $\|\sigma(E)y_n\|$.

The companion matrix formulation of this difference equation is

$$Y_{n+1} = \begin{pmatrix} 0 & I & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ c_1 & c_2 & c_3 & c_4 & \cdots & c_k \end{pmatrix} Y_n + P_n$$

$$\equiv C_n Y_n + P_n$$

where

$$c_j = -(\alpha_k I - h\beta_k J_n)^{-1}(\alpha_{j-1} I - h\beta_{j-1} J_n)$$

$$P_n = \begin{pmatrix} 0 \\ \\ 0 \\ (\alpha_k I - h\beta_k J_n)^{-1} p_n \end{pmatrix}.$$

Suppose that there exists a matrix $G^*(\theta)$ as in section 3 such that $||Y_{n+1}||^2_{G^*(\theta)} \leq (\theta')^2 ||Y_n||^2_{G^*(\theta)}$ in the homogeneous case $p_n = 0$. Then it follows that in the inhomogeneous case, we have

$$||Y_{n+1}||_{G^*(\theta)} \leq ||C_n||_{G^*(\theta)} ||Y_n||_{G^*(\theta)} + \sqrt{g^*_{kk}(\theta)}\, ||(\alpha_k I - h\beta_k J_n)^{-1} p_n||$$
$$\leq \theta' ||Y_n||_{G^*(\theta)} + \sqrt{g^*_{kk}(\theta)}\, ||(\alpha_k I - h\beta_k J_n)^{-1} p_n||$$

where $g^*_{kk}(\theta)$ is the (k,k) element of $G^*(\theta)$. Applying this bound recursively gives

$$||Y_n||_{G^*(\theta)} \leq (\theta')^n ||Y_0||_{G^*(\theta)} + \sqrt{g^*_{kk}(\theta)} \sum_{\nu=1}^{n} (\theta')^{n-\nu} ||(\alpha_k I - h\beta_k J_{\nu-1})^{-1} p_{\nu-1}||. \tag{5.1}$$

In order to bound $||\sigma(E)y_n||$, note that $\rho(E)y_n = hJ_n\sigma(E)y_n + p_n$ leads to

$$\sigma(E)\, y_n - \frac{\beta_k}{\alpha_k} hJ_n\sigma(E)y_n = \left(\sigma(E) - \frac{\beta_k}{\alpha_k}\rho(E)\right)y_n - \frac{\beta_k}{\alpha_k}p_n.$$

So

$$||\sigma(E)y_n|| \leq \left\|\left(I - \frac{\beta_k}{\alpha_k}J_n h\right)^{-1}\right\| \left(\left\|\left(\sigma(E) - \frac{\beta_k}{\alpha_k}\rho(E)\right)y_n\right\| + \left\|\frac{\beta_k}{\alpha_k}p_n\right\|\right). \tag{5.2}$$

Let

$$\lambda = \max_{z_0, z_1, \ldots, z_{k-1}} \frac{\left\|\left(\sigma(E) - \frac{\beta_k}{\alpha_k}\rho(E)\right)z_0\right\|^2}{||Z_0||^2_{G^*(\theta)}},$$

so that

$$\left\|\left(\sigma(E) - \frac{\beta_k}{\alpha_k}\rho(E)\right)y_n\right\| \leq \sqrt{\lambda}\, ||Y_n||_{G^*(\theta)}. \tag{5.3}$$

. It follows from Lemma 3.4 of [1] that it is sufficient to consider the one-dimensional case in the determination of λ, that is we need only consider scaler z_j. Hence λ can be found by solving a generalized eigenvalue problem of the same form as the one used to compute b(O).

Combining (5.2) and (5.3) gives

$$||\sigma(E)y_n|| \leq \left\|\left(I - \frac{\beta_k}{\alpha_k}J_n h\right)^{-1}\right\| \left(\sqrt{\lambda}\, ||Y_n||_{G^*(\theta)} + ||\frac{\beta_k}{\alpha_k}p_n||\right).$$

Using (5.1) then gives

$$\begin{aligned}\|\sigma(E)y_n\| \leq & \left\|I - \frac{\beta_k}{\alpha_k}J_n h)^{-1}\right\| \Big(\sqrt{\lambda}\ (\theta')^n \ \| Y_0\|_{G^*(\theta)} \\ & + \sqrt{\lambda g^*_{kk}(\theta)} \sum_{\nu=1}^{n} (\theta')^{n-\nu} \ \| (\alpha_k I - h\beta_k J_{\nu-1})^{-1} p_{\nu-1} \| \\ & + \left\| \frac{\beta_k}{\alpha_k} p_n \right\| \Big) \\ \leq & \sqrt{\lambda g^*_{kk}(\theta)} \left\| (I - \frac{\beta_k}{\alpha_k} J_n h)^{-1} \right\| \Big(\frac{(\theta')^n \| Y_0 \|_{G^*(\theta)}}{\sqrt{g^*_{kk}(\theta)}} \\ & + \sum_{\nu=1}^{n} (\theta')^{n-\nu} \| (\alpha_k I - h\beta_k J_{\nu-1})^{-1} p_{\nu-1} \| \\ & + \frac{\| \frac{\beta_k}{\alpha_k} p_n \|}{\sqrt{\lambda g^*_{kk}(\theta)}} \Big).\end{aligned}$$

Because of this bound, the quantity

$$c(\theta) = \sqrt{\lambda g^*_{kk}(\theta)}$$

is of interest and has been tabulated in section 7. Because of the form of the bound, it may be that the quantity $c(\theta)/\alpha_k$ is even more interesting.

6. Checking the algorithm.

It is interesting to test the matrix G constructed numerically by checking to see whether points in the stability region of the corresponding method do indeed lead to bounded solutions in the G-norm for the linear test equation $y' = \lambda y$. This is a reassuring test of both the theory and the implementation of the algorithm I'. For this test equation, the method (2) becomes

$$\rho(E)y_n = h\lambda\sigma(E)y_n,$$

which can be rewritten as

$$\rho(E)y_n - q\sigma(E)y_n = 0$$

where $q = h\lambda$. Letting the polynomial $\phi = \rho - q\sigma$ gives

$$\phi_0 y_n + \phi_1 y_{n+1} + \cdots + \phi_k y_{n+k} = 0.$$

This equation can be solved for y_{n+k}, yielding

$$y_{n+k} = -\frac{\phi_0}{\phi_k} y_n - \frac{\phi_1}{\phi_k} y_{n+1} - \cdots - \frac{\phi_{k-1}}{\phi_k} y_{n+k-1},$$

which can be employed to give the following matrix equation

$$\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+k-1} \\ y_{n+k} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\frac{\phi_0}{\phi_k} & -\frac{\phi_1}{\phi_k} & -\frac{\phi_2}{\phi_k} & -\frac{\phi_3}{\phi_k} & \cdots & -\frac{\phi_{k-1}}{\phi_k} \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{n+k-2} \\ y_{n+k-1} \end{pmatrix}$$

If we denote the above companion matrix by C_ϕ, we have the relationship $Y_{n+1} = C_\phi Y_n$. Hence, stability means that

$$||C_\phi Y_n||_G \leq ||Y_n||_G,$$

i.e., that $||C_\phi||_G \leq 1$. Thus we would expect $|\lambda| \leq 1$ for all solutions λ of the generalized eigenvalue problem

$$C_\phi^H G C_\phi z = \lambda G z.$$

This should hold for all values of q in the stability region, Recall here that $\phi = \rho - q\sigma$. In practice we have solved this eigenvalue problem for values of q lying on the boundary of various "disks" $\{q\text{: Re } \left(\frac{aq+b}{cq+d}\right) \leq m\}$ which should lie in $S(\rho, \sigma)$. For example, for the disk $\{q\text{: Re } (1/q) \leq m\}$, we calculate max $|\lambda|$ at the points $q_1 = 0$, $q_2 = 1/m$, and $q_3 = (m-i)/(m^2+1)$. If max $|\lambda| \leq 1$ at q_1, q_2, and q_3, then by other considerations we know that max $|\lambda| \leq 1$ in the convex hull of $\{q_1, q_2, q_3, \bar{q}_3\}$. This is the region shown in figure 2.

All experiments have indeed given max $|\lambda| \leq 1$. Furthermore, as expected, the result max $|\lambda| = 1$ was found whenever the point q was actually on the boundary of $S(\rho, \sigma)$.

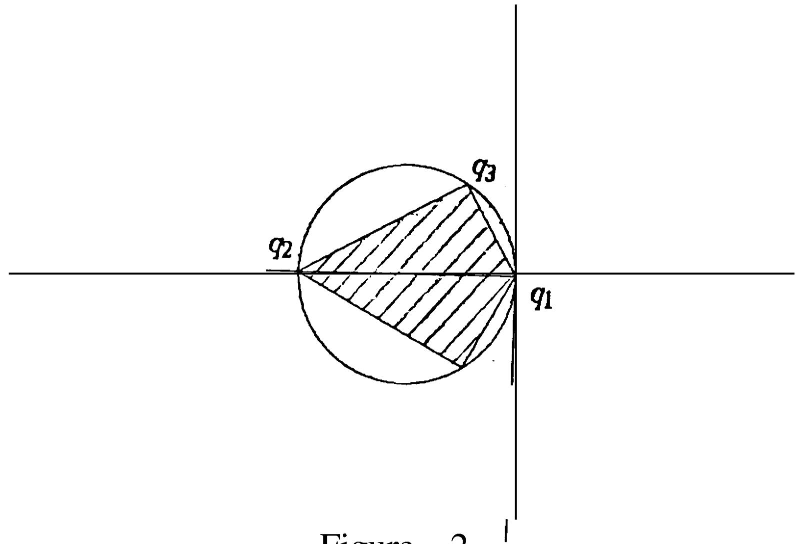


Figure 2

7. Numerical results,

The quantities $m(\theta)$, b(B), c(B), and n(M) have been calculated for the backwards differentiation methods, the Adams-Moulton methods, and the Adams-Bashforth methods of low order. In each case, the calculations were done for various values of the stepnumber k and for θ ranging from 1.3 down to 1.0. In the case of half-plane stability regions for the backwards differentiation methods, values of θ less than 1.0 were also allowed,

We recall that

$m(\theta)$ is defined on page 6, see also fig. 1 on page 7.

b(B) is the factor occurring in the expression on page 6 for the growth factor θ', when $\mu h > m(\theta)$.

c(0) is defined on page 10. It relates a weighted sum of the local errors to the global error for the linear multistep method.

$\kappa(M)$ is the Euclidean condition number of the transformation from Euclidean to G-norm, see page 3.

The coefficients of p and a for each method can be found in [3] or [4], for example. The tests were all run on the modified polynomials ρ^{**} and σ^{**} defined by (3.5).

All calculations were performed on an IBM 370 computer using double precision.

For $k = 2$ and $k = 3$, we have also plotted θ' as a function of θ for various values of μh, according to the definition (3.7). These functions are approximated by piecewise polynomials interpolating the values of θ given in the tables, Backwards Differentiation Methods, half-plane case

θ		k=2	k=3	k=4	k=5	k=6
1.3	m(θ)	0.257	0.261	0.211	-0.255	-1.436
	b(θ)	1.457	1.186	1.096	0.861	0.606
	c(θ)	0.996	1.055	0.994	0.960	1.076
	κ(M)	6.291	32.70	82.41	246.8	814.0
1.2	m(θ)	0.181	0.182	0.021	-0.698	-2.401
	b(θ)	1.397	1.176	1.015	0.749	0.497
	c(θ)	1.021	1.070	0.950	0.971	1.154
	κ(M)	6.064	28.33	62.48	.83.7	583.5
1.1	m(θ)	0.095	0.081	-0.256	-1.347	-3.840
	b(θ)	1.352	1.158	0.910	0.629	0.392
	c(θ)	1.057	1.027	0.927	1.007	1.278
	κ(M)	5.897	20.42	47.81	36.1	412.4
1.0	m(θ)	0.000	-0.083	-0.667	-2.327	-6.075
	b(θ)	1.333	1.072	0.788	0.506	0.296
	c(θ)	1.111	0.959	0.923	1.077	1.471
	κ(M)	5.828	14.80	36.62	99.88	286.5
.9	m(θ)	-0.117	-0.330	-1.293	-3.868	-9.719
	b(θ)	1.237	0.959	0.653	0.388	0.211
	c(θ)	1.047	0.916	0.945	1. 19'8	1.774
	κ(M)	5.042	11.47	27.92	72.31	195.0

θ		k=2	k=3	k=4	k=5	k=6
.8	m(θ)	-0.281	-0.708	-2.289	-6.426	-16.05
	b(θ)	1.123	0.824	0.513	0.279	0.141
	c(θ)	0.989	0.895	1.004	1.400	2.267
	κ(M)	4.329	9.046	21.10	51.46	130.0
.7	m(θ)	-0.520	-1.311	-3.969	-10.98	-27.96
	b(θ)	0.990	0.672	0.377	0.186	0.086
	c(θ)	0.940	0.899	1.120	1.745	3.115
	κ(M)	-- 3.689	7.161	15.75	35.87	83.69
.6	m(θ)	-0.889	-2.338	-7.031	-19.86	-52.96
	b(θ)	0.837	0.512	0.254	0.112	0.047
	c(θ)	0.904	0.942	1.337	2.363	4.705
	κ(M)	3.120	5.648	11.57	24.39	52.31
.5	m(θ)	-1.500	-4.245	-13.26	-39.63	-114.0
	b(θ)	0.667	0.354	0.152	0.059	0.022
	c(θ)	0.889	1.051	1.752	3.575	8.076
	κ(M)	2.618	4.418	8.338	16.09	31.45
.4	m(θ)	-2.625	-8.279	-28.10	-92.81	-300.3
	b(θ)	0.485	0.214	0.078	0.026	0.008
	c(θ)	0.911	1.292	2.616	6.331	16.71
	κ(M)	2.182	3.414	5.864	10.25	18.06

θ		k=2	k=3	k=4	k=5	k=6
.3	m(θ)	-5.056	-18.74	-73.65'	-286.7	-1111
	b(θ)	0.305	0.106	0.031	0.008	0.002
	c(θ)	1.007	1.847	4.772	14.36	46.70
	κ(M)	1.806	2.596	4.005	6.252	9.806
.2	m(θ)	-12.00	-57.85	-294.7	-1519	-7928
	b(θ)	0.148	0.036	0.008	0.002	3.0×10^{-3}
	c(θ)	1.289	3.428	12.40	51.34	229.1
	κ(M)	1.488	1.937	2.640	3.622	6.513
.1	m(θ)	-49.50	-403.9	-3535	-3.2×10^{4}	-3.0×10^{5}
	b(θ)	0.039	0.005	0.611×10^{-3}	6.8×10^{-5}	7.3×10^{-6}
	c(θ)	2.311	11.57	76.00	568.1	4588
	κ(M)	1.221	1.412	1.667	1.973	2.338

Backwards Differentiation Methods, disk case

θ		k=3	k=4	k=5	k=6
L.3	m(θ)	0.171	0.062	-0.044	-0.173
	b(θ)	5.493	5.013	4.454	3.938
	c(θ)	0.896	0.924	0.978	1.055
	κ(M)	19.07	70.71	274.9	1212
L.2	m(θ)	0.128	6.024×10^{-3}	-0.117	-0.288
	b(θ)	4.989	4.444	3.935	3.517
	c(θ)	0.895	0.944	1.016	1.111
	κ(M)	16.95	61.62	247.0	1150
..1	m(θ)	0.059	-0.073	-0.218	-0.477
	b(θ)	4.307	3.830	3.462	3.182
	c(θ)	0.930	1.000	1.084	1.192
	κ(M)	16.09	56.41	243.7	1028
.0	m(θ)	-0.071	-0.183	-0.368	-0.893
	b(θ)	3.365	3.273	3.064	2.820
	c(θ)	1.108	1.120	1.175	1.274
	κ(M)	20.96	54.36	162.4	526.1

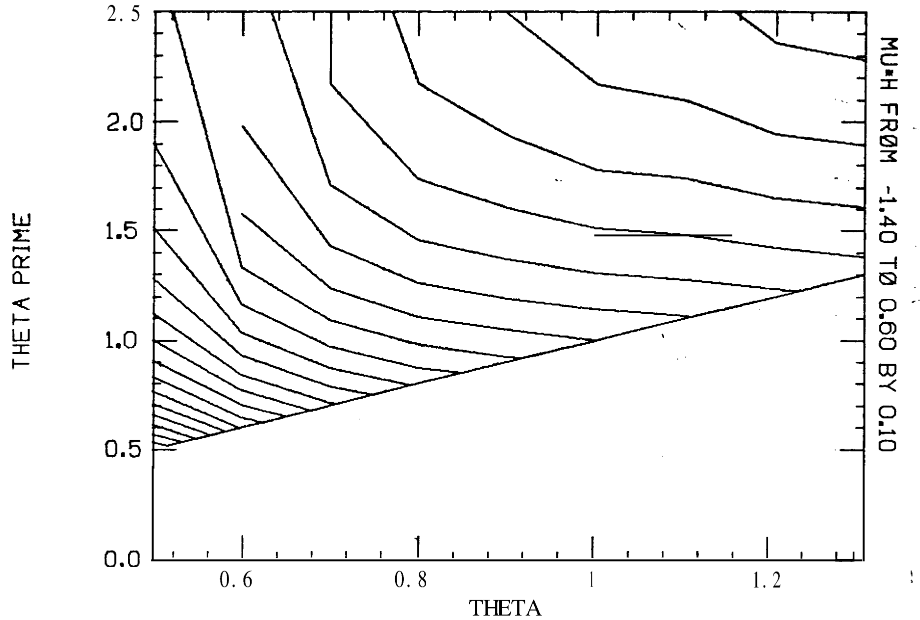
Adams-Bashforth Methods, disk case

θ		k=2	k=3	k=4	k=5	k=6
1.3	m(θ)	-0.819	-1.386	-2.289	-3.774	-6.261
	b(θ)	2.304	1.587	1.176	0.882	0.545
	κ(M)	2.964	8.812	35.93	132.0	390.8
1.2	m(θ)	-0.871	-1.508	-2.558	-4.348	-7.454
	b(θ)	2.202	1.484	1.083	0.787	0.503
	κ(M)	2.774	7.888	31.27	104.0	289.0
1.1	m(θ)	-0,931	-1.654	-2.896	-5.101	-9.085
	b(θ)	2.100	1.379	1.002	0.697	0.440
	κ(M)	2.590	7.085	27.65	'81.93	216.0
1.0	m(θ)	-1.000	-1.833	-3.33	-6.122	-11.40
	b(θ)	2.000	1.273	0.806	0.482	0.272
	κ(M)	2.414	6.427	18.14	52.93	155.6

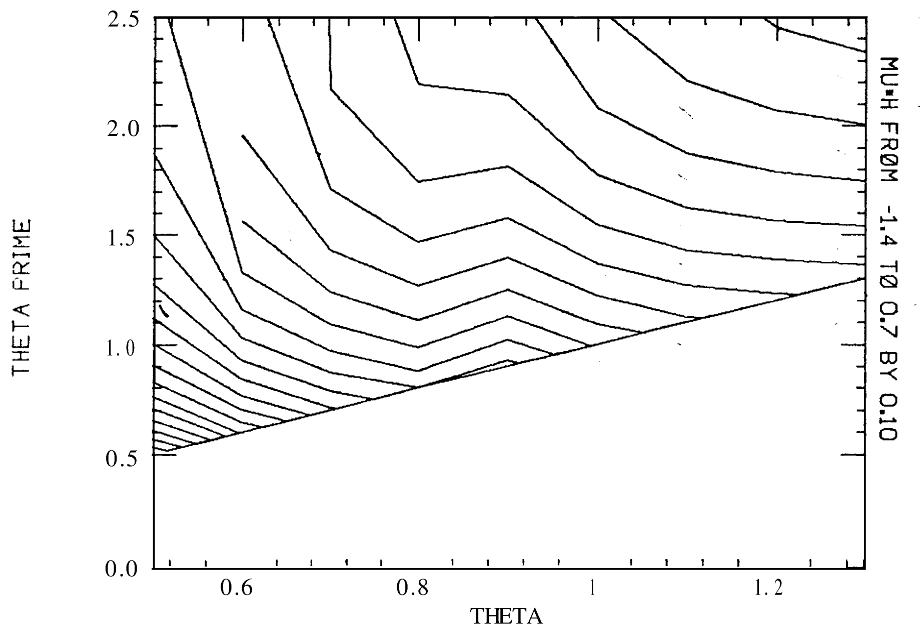
Adams-Moulton Methods, disk case

θ		k=2	k=3	k=4	k=5
1.3	m(θ)	-0.082	-0.213	-0.359	-0.542
	b(θ)	3.654	3.247	2.925	2.660
	c(θ)	1.083	1.172	1.267	1.379
	κ(M)	4.896	22.35	84.19	338.7
1.2	m(θ)	-0.107	-0.247	-0.410	-0.622
	b(θ)	3.563	3.143	2.814	2.544
	c(θ)	1.083	1.173	1.275	1.401
	κ(M)	4.523	19.14	68.55	276.0
1.1	m(θ)	-0.135	-0.287	-0.471	-0.720
	b(θ)	3.487	3.048	2.711	2.438
	c(θ)	1.083	1.176	1.286	1.432
	κ(M)	4.149	16.21	55.50	233.4
1.0	m(θ)	-0.167	-0.333	-0.544	-0.844
	b(θ)	3.429	2.963	2.563	2.167
	c(θ)	1.083	1.181	1.301	1.468
	κ(M)	3.777	13.54	37.94	107.0

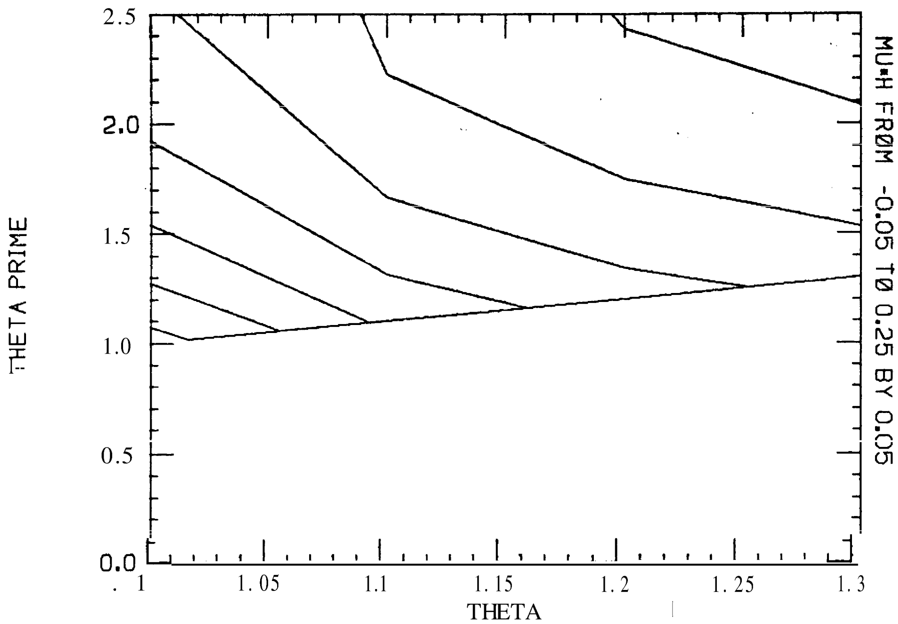
2-step Backwards Differentiation Method, half-plane case ,



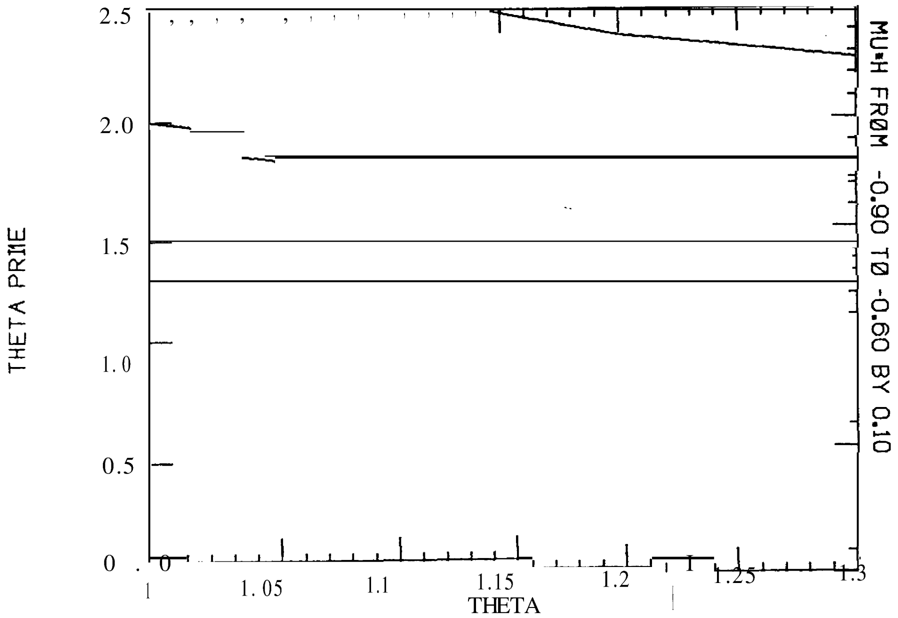
3-step Backwards Differentiation Method, half-plane case



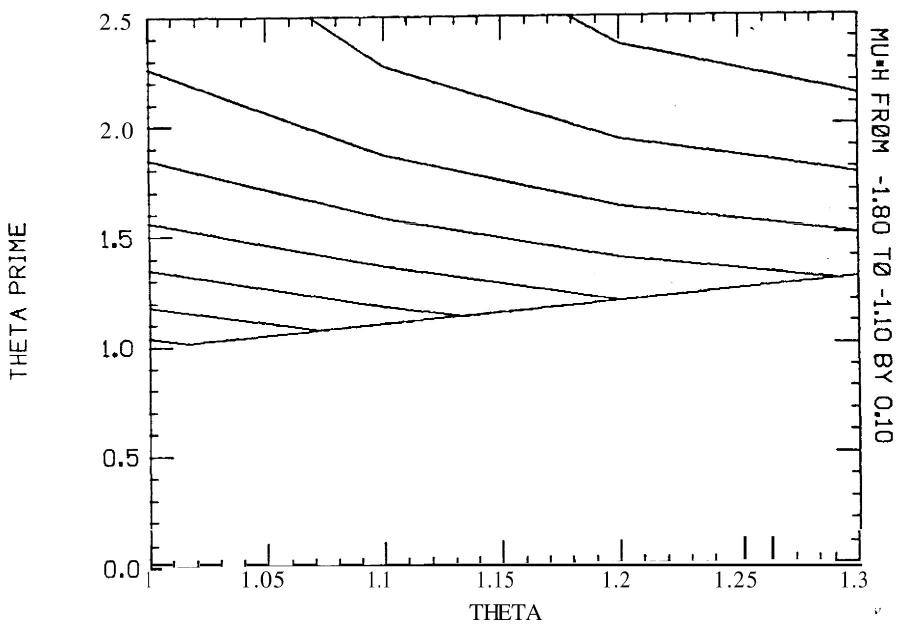
3-step Backwards Differentiation Method, disk case



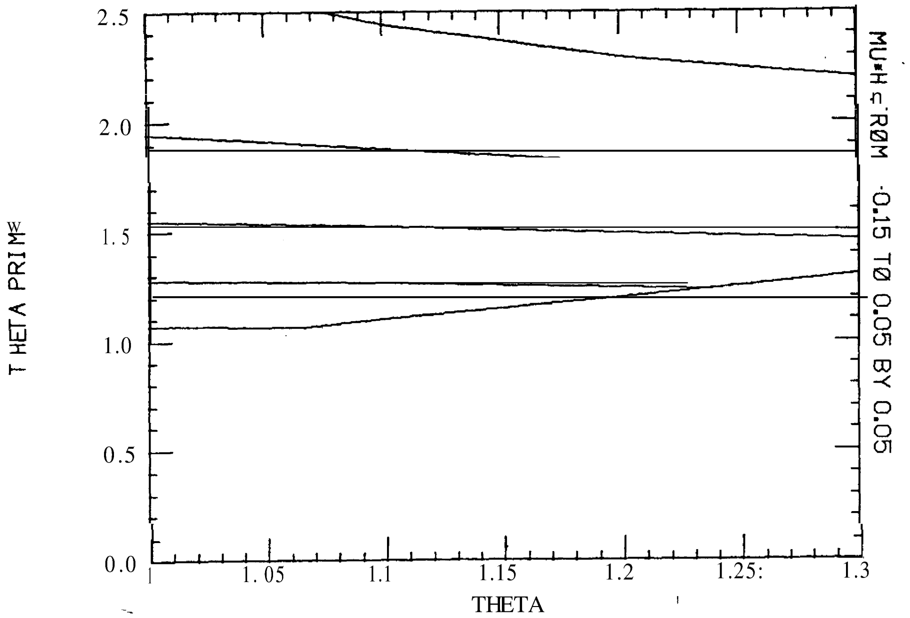
2-step Adams-Bashforth Method, disk case



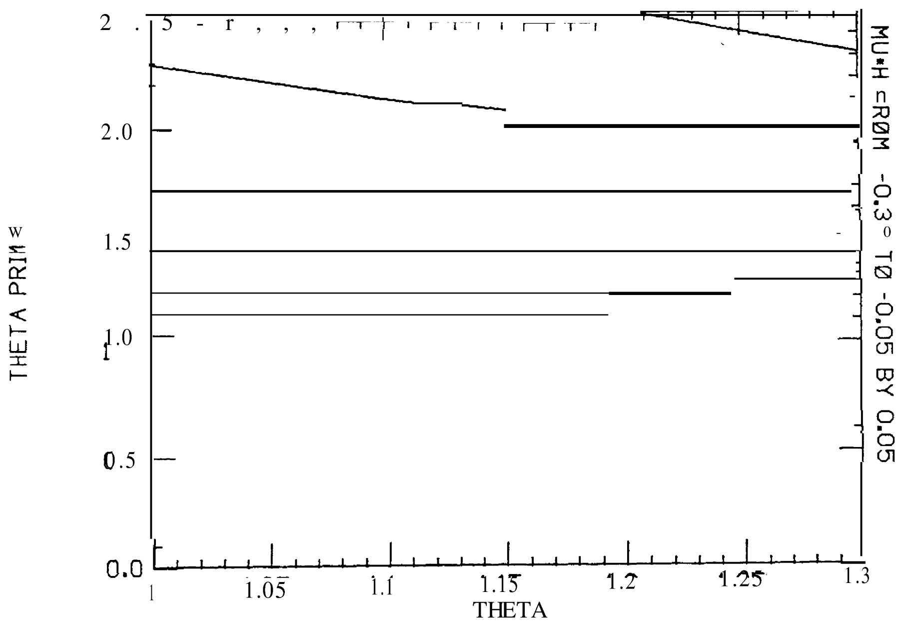
3-step Adams-Bashforth Method, disk case



2-step Adams-Moulton Method, disk case



3-step Adams-Moulton Method, disk case



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